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Extension Theorems for Spaces Arising from Approximation by Translates of a Basic Function

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We establish extension theorems for functions in spaces which arise naturally in studying interpolation by radial basic functions. These spaces are akin in some way to the non-integer-valued Sobolev spaces, although they are considerably more general. Such extensions allow us to establish local error estimates in a way which we make precise in the introductory section of our paper. There are many other applications of these fundamental results, including improved L_p error estimates for interpolation by shifts of a single basic function, but these applications have been left to a later paper.

1. INTRODUCTION

An interpolation problem using translates of a radial basic function takes the following form. Interpolation data are given consisting of distinct points $x_1, \ldots, x_m \in \mathbb{R}^n$ and corresponding values $d_1, \ldots, d_m \in \mathbb{R}$. We wish to interpolate these data by a function of the form

$$s(x) = \sum_{i=1}^{m} \mu_i \phi(|x - x_i|) + \sum_{j=1}^{\ell} v_j p_j(x). \tag{1}$$

The notation used here is as follows. The function ϕ is real-valued on $\mathbb{R}_+ = \{y \in \mathbb{R} : y \geqslant 0\}$, and $|\cdot|$ denotes the Euclidean norm. The functions $p_1, \ldots, p_l$ form a basis for Π_{k-1}, the space of polynomials on $\mathbb{R}^n$ whose total

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degree is at most $k-1$. Thus ℓ is the dimension of Π_{k-1}. The parameters $\mu_1, \ldots, \mu_m$ and $\nu_1, \ldots, \nu_\ell$ are real numbers to be determined by the equations

$$s(x_j) = d_j, \qquad j = 1, \ldots, m,$$

and

$$\sum_{i=1}^{m} \mu_i p_j(x_i) = 0, \qquad j = 1, \ldots, \ell.$$

The first condition ensures that our chosen s interpolates the data. The second is included to allow us to obtain a unique solution to the system, provided the points $x_1, \ldots, x_m$, the parameter k, and the function ϕ satisfy certain conditions. First, we ask that ϕ be strictly conditionally positive definite of order k, by which we mean the following. If, for any set of distinct points $x_1, x_2, \ldots, x_r \in \mathbb{R}^n$ and constants $c_1, c_2, \ldots, c_r$ (not all of which are zero), the quadratic form

$$\sum_{i=1}^{r} \sum_{j=1}^{r} c_i c_j \phi(|x_i - x_j|) > 0$$

whenever

$$\sum_{i=1}^{r} c_i p(x_i) = 0 \qquad \text{for all} \quad p \in \Pi_{k-1},$$

then we say ϕ is strictly conditionally positive definite of order k. Our second requirement for a unique solution is that $x_1, \ldots, x_m$ be unisolvent with respect to Π_{k-1}. That is, if $q \in \Pi_{k-1}$ satisfies $q(x_j) = 0$ for all $j = 1, \ldots, m$ then q must be the zero polynomial. What is important to note here is that in many common choices of ϕ, k is at most 2. Therefore at worst we are adding a linear polynomial to our interpolant and often no polynomial part is required. Thus the unisolvency condition is not as problematic as it seems at first glance. Some examples of ϕ with the value of k needed to obtain a unique solution are listed below:

Bare norm	$\phi(r) = r$	$k = 0$
Thin plate spline	$\phi(r) = r^2 \ln r$	$k = 2$
Multiquadric	$\phi(r) = \sqrt{r^2 + c^2}$	$k = 0$
Gaussian	$\phi(r) = e^{-r^2}$	$k = 0$

Notice that in three of these cases $k = 0$, indicating that no polynomial part is used in the interpolant.

Duchon [2] was the first to look at these types of interpolation problems and used a variational approach. He was interested in surface splines where the radial function takes the form $\phi(r)=r^\lambda \ln r$ or $\phi(r)=r^\lambda$ for $r\geqslant 0$. Simple examples of these are the bare norm and thin plate spline. These were shown by Duchon [2] to be the natural multivariate analogue of natural splines.

Later work focused on the solvability of the interpolation problem and its reliance on the notion of conditionally positive definite functions. Inspired by the numerical results of Hardy [4], the seminal paper by Micchelli [13] proved, amongst other things, that the multiquadric interpolation problem was always solvable.

Some powerful results have been achieved by employing both of these ideas. One begins with a conditionally positive definite function and builds around it a *native space* in which one can carry out variational arguments. Fundamental papers in this area are those of Madych and Nelson [11, 12], Wu and Schaback [18] and several papers by Schaback which are accessible through the survey [15].

We return now to Duchon's variational approach. The interpolant is shown to be a minimal norm interpolant in the following sense. One has a space of functions X and a seminorm $|\cdot|$ defined on X. Given $f\in X$ we wish to find $s\in X$ such that

$$\begin{aligned} &\text{(i)}\quad s(x_j)=f(x_j), \qquad \text{for all} \quad j=1,\ldots,m,\\ &\text{(ii)}\quad |s|\leqslant|v|, \qquad \text{for all } v\in X \text{ satisfying } v(x_j)=f(x_j) \text{ for all } j=1,\ldots,m. \end{aligned} \tag{2}$$

The function s is known as the minimal norm interpolant to f on $x_1,\ldots,x_m$. A useful result concerning minimal norm interpolants is that $|f-s|^2=|f|^2-|s|^2$. We shall make use of this at the end of the section.

Duchon used spaces of distributions which were generalisations of Beppo–Levi spaces. We shall be interested in the function spaces introduced by Light and Wayne in [10]. A measurable weight function v is introduced and the seminorm is defined, for $k\in\mathbb{Z}_+$, as follows:

$$|f|=\left(\sum_{|\alpha|=k} c_\alpha \int_{\mathbb{R}^n} |\widehat{D^\alpha f}(x)|^2\, v(x)\,dx\right)^{1/2}.$$

The constants c_α are chosen so as to make the seminorm radially symmetric, whenever v is radially symmetric. The Fourier transform is taken in the distributional sense. The space of functions is given by

$$Z(\mathbb{R}^n)=\{f\in S' : \widehat{D^\alpha f}\in L^1_{loc}(\mathbb{R}^n) \text{ for all } \alpha\in\mathbb{Z}^n_+ \text{ with } |\alpha|=k \text{ and } |f|<\infty\}.$$

Here we use S' to denote the space of tempered distributions. Light and Wayne demonstrated that, for suitable choices of the weight function v, and for suitable values of k, $Z(\mathbb{R}^n)$ was embedded in $C(\mathbb{R}^n)$, and thus point evaluations made sense.

We turn our attention now to error estimates on a space X with seminorm $|\cdot|$. We interpolate a function $f \in X$ at points $x_1, \ldots, x_m$ by $s \in X$. A typical error estimate has the form

$$|f(x)-s(x)| \leqslant \mathscr{P}(x, x_1, \ldots, x_m)\, |f-s|, \qquad \text{for all} \quad x \in \mathbb{R}^n.$$

Here $\mathscr{P}$ is the so-called power function whose form can be explicitly obtained. In order to be able to use this we need to know $f-s$ everywhere on $\mathbb{R}^n$. Duchon explained why it would be useful to be able to obtain for $\Omega \subset \mathbb{R}^n$ a "local" estimate of the form

$$|f(x)-s(x)| \leqslant \mathscr{P}(x, x_1, \ldots, x_m)\, |f-s|_\Omega, \qquad \text{for all} \quad x \in \Omega. \tag{3}$$

We notice that a localised version of the seminorm appears on the right-hand side and the error estimate is now only true for $x \in \Omega$. Using this one can deduce improved L_p error estimates in terms of the spacing of the interpolation points $x_1, \ldots, x_m$. Let $A = \{x_1, \ldots, x_m\}$ and

$$h = \max_{y \in \Omega} \min_{x \in A} |y-x|.$$

Suppose that using the original error estimate one can obtain a constant C independent of f and h such that $\|f-s\|_{p,\Omega} \leqslant Ch^\beta\, |f|$ for some β. Making use of the localised error estimate it is possible to improve this to $\|f-s\|_{p,\Omega} \leqslant Ch^{\beta+n/p}\, |f|_\Omega$. The exact details of this can be found in Duchon [3], or the later work of Light and Wayne [9]. To obtain these improved results we must first derive the local error estimate (3). In order to do this we need two ingredients.

First, we need to explain what we mean by the local seminorm $|\cdot|_\Omega$. Recall that our seminorm is defined in terms of the Fourier transform of the function. Thus there is currently no natural way of defining the local seminorm. What is needed is a direct version of the seminorm, defined in terms of the function itself, and not its Fourier transform. The recent paper of Levesley and Light [8] concerned itself with this task. They were able to prove, again with certain assumptions on the weight function v, that for all $f \in X(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} |\widehat{D^\alpha f}(x)|^2\, v(x)\, dx = -\tfrac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{v}(x-y)\, |(D^\alpha f)(x) - (D^\alpha f)(y)|^2\, dx\, dy.$$

Now we can simply define our local seminorm by

$$|f|_\Omega = \left(\sum_{|\alpha| = k} c_\alpha \int_\Omega \int_\Omega w(x-y)\, |(D^\alpha f)(x) - (D^\alpha f)(y)|^2 \, dx \, dy \right)^{1/2},$$

where $w = -\frac{1}{2}\hat{v}$.

The second requirement for the development of the localised version of the error estimate is having certain extension operators to hand. Duchon worked in a Sobolev space setting where the relevant extension theorems were already well known. The development of the required extension operators for the above seminorm is the aim of our current research. The reader may have noticed the resemblance of the direct seminorm to that used in non-integer-valued Sobolev spaces. The extension theorems for these spaces serve as a guide to the development of our theory, although at the level of generality considered in this paper, significant new techniques are needed. We begin by working with some spaces of continuous functions. For $k \in \mathbb{Z}_+$, let $C_0^k(\mathbb{R}^n)$ be the set of all compactly supported functions on $\mathbb{R}^n$ which have continuous derivatives up to the kth order. As usual $C_0^\infty(\mathbb{R}^n)$ will denote $\bigcap_{k=0}^\infty C_0^k(\mathbb{R}^n)$. Let Ω be an open subset of $\mathbb{R}^n$. The space of functions we shall initially be interested in is $X(\Omega) = \{g|_\Omega \colon g \in C_0^k(\mathbb{R}^n) \text{ and } |g|_\Omega < \infty\}$. Similarly we define $X(\mathbb{R}^n) = \{f \in C_0^k(\mathbb{R}^n) : |f|_{\mathbb{R}^n} < \infty\}$. Now, under appropriate hypotheses on w, $|\cdot|_\Omega$ defines a seminorm on $X(\Omega)$. It will be of use to define a norm on $X(\Omega)$ as follows:

$$\|f\|_\Omega = \left(\sum_{|\alpha| \leqslant k} \int_\Omega |D^\alpha f(x)|^2 \, dx + |f|_\Omega^2 \right)^{1/2}.$$

The norm $\|\cdot\|_{\mathbb{R}^n}$ on $X(\mathbb{R}^n)$ is defined similarly. We develop a linear extension operator from $X(\Omega)$ to $X(\mathbb{R}^n)$, subject to Ω and the weight function w satisfying certain properties which are detailed later. Using this result we deduce the existence of extensions for functions in $\mathscr{X}(\Omega)$, the completion of $X(\Omega)$ with respect to the norm $\|\cdot\|_\Omega$. Outlined below is our principal result, the proof of which is again dependent on a suitable choice of Ω and w.

THEOREM 1.1. *Given $f \in \mathscr{X}(\Omega)$, there exists a function $f_e \in \mathscr{X}(\mathbb{R}^n)$ such that*

(1) $f_e|_\Omega = f$

(2) $|f_e|_{\mathbb{R}^n} \leqslant M\, |f|_\Omega$ *for some constant M independent of f.*

We now give a demonstration of how these extension theorems can be used in the development of improved error estimates. The example is chosen because it needs no detailed understanding of the power function $\mathscr{P}$. Let $X(\Omega)$ be a space of functions on Ω with seminorm $|\cdot|_\Omega$. Suppose $x_1, ..., x_m$ lie in $\Omega \subset \mathbb{R}^n$. Take $f \in X(\Omega)$ and let s_f be the minimal norm interpolant to f on $x_1, \ldots, x_m$. Suppose we can find $f_e \in X(\mathbb{R}^n)$ such that $f_e|_\Omega = f|_\Omega$ and $|f_e|_{\mathbb{R}^n} \leqslant C\,|f|_\Omega$ for some constant C independent of f. Let s_{f_e} be the minimal norm interpolant to f_e on $x_1, \ldots, x_m$. The usual error estimate takes the form

$$|f_e(x) - s_{f_e}(x)| \leqslant \mathscr{P}(x, x_1, \ldots, x_m)\,|f_e - s_{f_e}|_{\mathbb{R}^n}.$$

Now s_{f_e} is the minimal norm interpolant to f_e and so

$$\begin{aligned}|f_e - s_{f_e}|^2_{\mathbb{R}^n} &= |f_e|^2_{\mathbb{R}^n} - |s_{f_e}|^2_{\mathbb{R}^n}\\ &\leqslant |f_e|^2_{\mathbb{R}^n}.\end{aligned}$$

Hence,

$$\begin{aligned}|f_e(x) - s_{f_e}(x)| &\leqslant \mathscr{P}(x, x_1, \ldots, x_m)\,|f_e|_{\mathbb{R}^n}\\ &\leqslant C\mathscr{P}(x, x_1, \ldots, x_m)\,|f|_\Omega.\end{aligned}$$

Since $x_1, \ldots, x_m$ lie in Ω we have $s_f = s_{f_e}$. Therefore, the previous inequality can be written as

$$|f_e(x) - s_f(x)| \leqslant C\mathscr{P}(x, x_1, \ldots, x_m)\,|f|_\Omega.$$

Then for all $x \in \Omega$ we have

$$|f(x) - s_f(x)| \leqslant \mathscr{P}(x, x_1, \ldots, x_m)\,C\,|f|_\Omega.$$

This paper has in our view two significant omissions. First, we do not go on from establishing the extension in Theorem 1.1 to the application we have indicated—that of obtaining improved error estimates. The arguments needed for these results are not trivial, and reasons of space ruled them out of this paper. Second, we have to make some mention of other work in this area. Extension theorems already exist in the work of Schaback [16] and Iske [6]. These papers also contain alternative versions of the space $X(\Omega)$ to that given by Levesley and Light [8]. At the time of writing, we have established that these alternative definitions of $X(\Omega)$ produce different spaces for certain choices of Ω. To be more precise, we have constructed an example of a domain Ω with an exponential cusp in its boundary for which the space constructed by Iske is smaller than the corresponding Levesley–Light space. Both Schaback and Iske produce an extension operator which is an isometry. Our extension is never an isometry, so again there is a fundamental difference. Finally, as we point out in Section 4, our theory includes the standard Sobolev theory as a special case. At the time of writing, we are unable to be more authoritative than this, but are currently exploring thoroughly the connections and distinctions between our work and that of Schaback and his co-authors.

We need to introduce some terminology, which will be adhered to throughout the paper. The space S is the space of rapidly decreasing functions endowed with the usual topology (Rudin [14]). For $f \in S$ the Fourier transform of f is defined by

$$\hat{f}(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(y)\, e^{-ixy}\, dy, \qquad x \in \mathbb{R}^n.$$

The space $L^1_{loc}(\mathbb{R}^n)$ is made up of all measurable functions $f\colon \mathbb{R}^n \to \mathbb{C}$ such that for any compact set K in $\mathbb{R}^n$, $f|_K \in L^1(K)$. The space of all polynomials of total degree at most k will be denoted by Π_k. For $\gamma \in \mathbb{Z}^n_+$, the operator D^γ is the usual (distributional) partial derivative of order γ. Finally, we have already in this Section overworked the notation $|\cdot|$. Sometimes this notation is used for a seminorm, and sometimes for the Euclidean norm of a point in $\mathbb{R}$ or $\mathbb{R}^n$. We believe that the intention is always clear from the context and that this economy of notation is more helpful than confusing.

2. AN EXTENSION ON $\mathbb{R}^n_+$

Our first extension is from $\mathbb{R}^n_+$ to the whole of $\mathbb{R}^n$. By $\mathbb{R}^n_+$, we mean the set of all points in $\mathbb{R}^n$ whose last coordinate is positive. Because of our focus on the last coordinate, it will help to write a point $x \in \mathbb{R}^n$ in the form $x = (x', x_n)$, where $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$. Then $\mathbb{R}^n_+ = \{(x', x_n)\colon x_n > 0\}$. For $k \in \mathbb{Z}_+$, we define $Y^k(\mathbb{R}^n_+) = \{g|_{\mathbb{R}^n_+} : g \in C^k_0(\mathbb{R}^n)\}$.

Definition 2.1. Let $k \in \mathbb{Z}_+$. Define $\lambda_1, ..., \lambda_{k+1}$ to be the unique solution of the system

$$\sum_{j=1}^{k+1} \lambda_j \left(-\frac{1}{j}\right)^l = 1, \qquad l = 0, 1, ..., k.$$

For each $f\colon \mathbb{R}^n_+ \to \mathbb{R}$ and each $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{Z}^n_+$, define $E_\alpha f\colon \mathbb{R}^n \to \mathbb{R}$ by

$$E_\alpha f(x', x_n) = \begin{cases} f(x', x_n), & \text{if } \ x_n > 0 \\ \displaystyle\sum_{j=1}^{k+1} \lambda_j \left(-\frac{1}{j}\right)^{|\alpha_n|} f(x', -x_n/j), & \text{otherwise.} \end{cases}$$

Theorem 2.2. *Let $\theta = (0, ..., 0)$ and let $f \in Y^k(\mathbb{R}^n_+)$, for some $k \in \mathbb{Z}_+$. Then $E_\theta f \in C^k_0(\mathbb{R}^n)$ and $D^\alpha E_\theta f = E_\alpha D^\alpha f$ for all $\alpha \in \mathbb{Z}^n_+$ with $|\alpha| \leqslant k$.*

Proof. Let $f \in Y^k(\mathbb{R}^n_+)$. Suppose $x = (x', x_n) \in \mathbb{R}^n$, with $x_n \leqslant 0$, and $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{Z}^n_+$ with $|\alpha| \leqslant k$. Then

$$D^\alpha E_\theta f(x', x_n) = \sum_{j=1}^{k+1} \lambda_j \left(-\frac{1}{j}\right)^{|\alpha_n|} D^\alpha f(x', -x_n/j) = E_\alpha D^\alpha f(x', x_n).$$

The relation $D^\alpha E_\theta f(x) = E_\alpha D^\alpha f(x)$ for $x = (x', x_n)$ with $x_n > 0$ is clear and so the formula $D^\alpha E_\theta f = E_\alpha D^\alpha f$ is established for all $f \in Y^k(\mathbb{R}^n_+)$. Now it is clear that if $g \in Y^0(\mathbb{R}^n_+)$, then $E_\alpha g \in C_0(\mathbb{R}^n)$. From this we deduce that $E_\theta f \in C^k_0(\mathbb{R}^n)$. ▮

We now introduce a weight function $w: \mathbb{R}^n \to \mathbb{R}$ which is a measurable function with the properties

($\mathscr{W}$1) $\int_A w > 0$ whenever A has positive measure;

($\mathscr{W}$2) there exists a constant $M > 0$ such that if $x = (x', x_n) \in \mathbb{R}^n$ and $y = (x', y_n) \in \mathbb{R}^n$ with $|x_n| \geqslant |y_n|$ then $w(x) \leqslant Mw(y)$.

We note that the results in this section require only that w satisfy ($\mathscr{W}$2) and $w(x) \geqslant 0$ for almost all $x \in \mathbb{R}^n$. However, we make the stronger assumption ($\mathscr{W}$1) as this will be required in Sections 3 and 4.

Now take $\alpha \in \mathbb{Z}^n_+$, and suppose $f \in Y^{|\alpha|}(\mathbb{R}^n_+)$. We define

$$|f|_{\alpha, \mathbb{R}^n_+} = \left(\int_{\mathbb{R}^n_+} \int_{\mathbb{R}^n_+} w(x-y)\, |D^\alpha f(x) - D^\alpha f(y)|^2 \, dx\, dy\right)^{1/2}.$$

Note that $|f|_{\alpha, \mathbb{R}^n_+}$ is an extended real-valued number. We denote by $X^\alpha(\mathbb{R}^n_+)$ the set of all $f \in Y^{|\alpha|}(\mathbb{R}^n_+)$ which satisfy $|f|_{\alpha, \mathbb{R}^n_+} < \infty$. The quantity $|f|_{\alpha, \mathbb{R}^n}$ and the corresponding set $X^\alpha(\mathbb{R}^n)$ are similarly defined.

THEOREM 2.3. *There exists a linear operator E: $X^\alpha(\mathbb{R}^n_+) \to X^\alpha(\mathbb{R}^n)$ such that*

(i) *for all $f \in X^\alpha(\mathbb{R}^n_+)$ and $x \in \mathbb{R}^n_+$, $Ef(x) = f(x)$.*

(ii) *there exists a constant $A > 0$ such that $|Ef|_{\alpha, \mathbb{R}^n} \leqslant A\, |f|_{\alpha, \mathbb{R}^n_+}$ for all $f \in X^\alpha(\mathbb{R}^n_+)$.*

Proof. Our claim is that a suitable choice for E is the one we have already defined prior to the theorem, $E = E_\theta$, providing $|\alpha| \leqslant k$. It follows immediately from the construction of E that $Ef(x) = f(x)$ for all $x \in \mathbb{R}^n_+$ and all $f \in X^\alpha(\mathbb{R}^n_+)$.

Let $f \in X^\alpha(\mathbb{R}^n_+)$. We consider

$$|Ef|^2_{\alpha, \mathbb{R}^n} = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} w(x-y)\, |D^\alpha Ef(x) - D^\alpha Ef(y)|^2 \, dx\, dy.$$

It will be convenient to define the measurable function z by

$$z(x, y) = w(x-y)\,|D^\alpha Ef(x) - D^\alpha Ef(y)|^2,$$

for almost all $x, y \in \mathbb{R}^n$. Furthermore, we let χ_{++} be the characteristic function of $\mathbb{R}^n_+ \times \mathbb{R}^n_+$, χ_{+-} be the characteristic function of $\mathbb{R}^n_+ \times (\mathbb{R}^n \setminus \mathbb{R}^n_+)$, and similarly for χ_{-+} and χ_{--}. Then

$$|Ef|^2_{\alpha, \mathbb{R}^n} = I_{++} + I_{+-} + I_{-+} + I_{--}$$

where, for example,

$$I_{-+} = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \chi_{-+}(x, y)\, z(x, y)\, dx\, dy.$$

Now

$$I_{-+} = \int_{\mathbb{R}^{n-1}} \int_0^\infty \int_{\mathbb{R}^{n-1}} \int_{-\infty}^0 w(x' - y', x_n - y_n)$$
$$\left| \left(\sum_{j=1}^{k+1} \lambda_j \left(-\frac{1}{j} \right)^{|\alpha_n|} D^\alpha f(x', -x_n/j) \right) - D^\alpha f(y', y_n) \right|^2 dx_n\, dx'\, dy_n\, dy'.$$

Recall that since $|\alpha_n| \leqslant k$ we have $\sum_{j=1}^{k+1} \lambda_j (-1/j)^{|\alpha_n|} = 1$. Using this fact and an application of the Cauchy–Schwarz inequality gives

$$\left| \left(\sum_{j=1}^{k+1} \lambda_j \left(-\frac{1}{j} \right)^{|\alpha_n|} D^\alpha f(x', -x_n/j) \right) - D^\alpha f(y', y_n) \right|^2$$
$$= \left| \sum_{j=1}^{k+1} \lambda_j \left(-\frac{1}{j} \right)^{|\alpha_n|} (D^\alpha f(x', -x_n/j) - D^\alpha f(y', y_n)) \right|^2$$
$$\leqslant \left(\sum_{j=1}^{k+1} \left| \lambda_j \left(-\frac{1}{j} \right)^{|\alpha_n|} \right|^2 \right) \left(\sum_{j=1}^{k+1} |D^\alpha f(x', -x_n/j) - D^\alpha f(y', y_n)|^2 \right).$$

Let $A_1 = \sum_{j=1}^{k+1} |\lambda_j|^2 \, (-\frac{1}{j})^{2\,|\alpha_n|}$. Then

$$I_{-+} \leqslant A_1 \sum_{j=1}^{k+1} \int_{\mathbb{R}^{n-1}} \int_0^\infty \int_{\mathbb{R}^{n-1}} \int_{-\infty}^0 w(x' - y', x_n - y_n)$$
$$|D^\alpha f(x', -x_n/j) - D^\alpha f(y', y_n)|^2\, dx_n\, dx'\, dy_n\, dy'.$$

Making the substitution $x_n = -js_n$ in the appropriate integral gives

$$I_{-+} \leqslant A_1 \sum_{j=1}^{k+1} j \int_{\mathbb{R}^{n-1}} \int_0^\infty \int_{\mathbb{R}^{n-1}} \int_0^\infty w(x' - y', -js_n - y_n)$$
$$|D^\alpha f(x', s_n) - D^\alpha f(y', y_n)|^2\, ds_n\, dx'\, dy_n\, dy'.$$

Since j, s_n, and y_n only take positive values,

$$|-js_n - y_n| = |js_n + y_n| = js_n + y_n \geqslant s_n + y_n \geqslant |s_n - y_n|.$$

Hence by ($\mathscr{W}$2), we can find a number $A_2 > 0$ such that

$$I_{-+} \leqslant A_2 \sum_{j=1}^{k+1} j \int_{\mathbb{R}^{n-1}} \int_0^\infty \int_{\mathbb{R}^{n-1}} \int_0^\infty w(x'-y', s_n - y_n)$$
$$|D^\alpha f(x', s_n) - D^\alpha f(y', y_n)|^2 \, ds_n \, dx' \, dy_n \, dy'.$$

Letting $A_3 = A_2 \sum_{j=1}^{k+1} j$ we obtain

$$I_{-+} \leqslant A_3 \int_{\mathbb{R}^n_+} \int_{\mathbb{R}^n_+} w(x-y) \, |D^\alpha f(x) - D^\alpha f(y)|^2 \, dx \, dy.$$

An almost identical argument furnishes the existence of a constant A_4 such that

$$I_{+-} \leqslant A_4 \int_{\mathbb{R}^n_+} \int_{\mathbb{R}^n_+} w(x-y) \, |D^\alpha f(x) - D^\alpha f(y)|^2 \, dx \, dy.$$

Now by reasoning very similar to above, we deduce the existence of $A_5 > 0$ such that

$$I_{--} = \int_{\mathbb{R}^{n-1}} \int_{-\infty}^0 \int_{\mathbb{R}^{n-1}} \int_{-\infty}^0 w(x'-y', x_n - y_n)$$
$$\left| \sum_{j=1}^{k+1} \lambda_j \left(-\frac{1}{j} \right)^{|\alpha_n|} (D^\alpha f(x', -x_n/j) - D^\alpha f(y', -y_n/j)) \right|^2 dx_n \, dx' \, dy_n \, dy'$$
$$\leqslant A_5 \sum_{j=1}^{k+1} \int_{\mathbb{R}^{n-1}} \int_{-\infty}^0 \int_{\mathbb{R}^{n-1}} \int_{-\infty}^0 w(x'-y', x_n - y_n)$$
$$|D^\alpha f(x', -x_n/j) - D^\alpha f(y', -y_n/j)|^2 \, dx_n \, dx' \, dy_n \, dy'.$$

The change of variables $x_n = -js_n$ and $y_n = -jt_n$ gives

$$I_{--} \leqslant A_5 \sum_{j=1}^{k+1} j^2 \int_{\mathbb{R}^{n-1}} \int_0^\infty \int_{\mathbb{R}^{n-1}} \int_0^\infty w(x'-y', jt_n - js_n)$$
$$|D^\alpha f(x', s_n) - D^\alpha f(y', t_n)|^2 \, ds_n \, dx' \, dt_n \, dy'.$$

Again, since s_n, t_n, and j take only positive values, we have

$$|jt_n - js_n| = j \, |s_n - t_n| \geqslant |s_n - t_n|,$$

and so an application of ($\mathscr{W}2$) furnishes a constant A_6 such that

$$I_{--} \leqslant A_6 \int_{\mathbb{R}^n_+} \int_{\mathbb{R}^n_+} w(x'-y', s_n-t_n)\, |D^\alpha f(x', s_n) - D^\alpha f(y', t_n)|^2 \, ds_n \, dx' \, dt_n \, dy'.$$

Finally, using

$$\begin{aligned} |Ef|^2_{\alpha, \mathbb{R}^n} &= I_{++} + I_{+-} + I_{-+} + I_{--} \\ &\leqslant (1 + A_3 + A_4 + A_6) \int_{\mathbb{R}^n_+} \int_{\mathbb{R}^n_+} w(x-y)\, |D^\alpha f(x) - D^\alpha f(y)|^2 \, dx \, dy, \end{aligned}$$

we obtain $|Ef|_{\alpha, \mathbb{R}^n} \leqslant \sqrt{1 + A_3 + A_4 + A_6}\, |f|_{\alpha, \mathbb{R}^n_+}$. ∎

The essential ingredient from Definition 2.1 and Theorem 2.2 is the operator E_θ, which we will henceforward abbreviate to E. We end this section with some results that will be of use later.

LEMMA 2.4. *Let $\theta = (0, ..., 0) \in \mathbb{Z}^n_+$. Let $k \in \mathbb{Z}_+$, and let $\alpha \in \mathbb{Z}^n_+$ satisfy $|\alpha| \leqslant k$. Given $f \in X^\alpha(\mathbb{R}^n_+)$ let $Ef = E_\theta f$ be as defined in Theorem 2.2. Then*

$$\int_{\mathbb{R}^n} |D^\alpha Ef(x)|^2 \, dx \leqslant C \int_{\mathbb{R}^n_+} |D^\alpha f(x)|^2 \, dx$$

for some constant C independent of f.

Proof. We can write

$$\begin{aligned} &\int_{\mathbb{R}^n} |D^\alpha Ef(x)|^2 \, dx \\ &\quad = \int_{\mathbb{R}^n_+} |D^\alpha Ef(x)|^2 \, dx + \int_{\mathbb{R}^n \setminus \mathbb{R}^n_+} |D^\alpha Ef(x)|^2 \, dx \\ &\quad = \int_{\mathbb{R}^n_+} |D^\alpha f(x)|^2 \, dx + \int_{\mathbb{R}^{n-1}} \int_{-\infty}^0 \left| \sum_{j=1}^{k+1} \lambda_j \left(-\frac{1}{j}\right)^{|\alpha_n|} D^\alpha f(x', -x_n/j) \right|^2 dx_n \, dx'. \end{aligned}$$

We consider the second integral. Let

$$I = \int_{\mathbb{R}^{n-1}} \int_{-\infty}^0 \left| \sum_{j=1}^{k+1} \lambda_j \left(-\frac{1}{j}\right)^{|\alpha_n|} D^\alpha f(x', -x_n/j) \right|^2 dx_n \, dx'.$$

An application of the Cauchy–Schwarz inequality gives

$$I \leqslant \int_{\mathbb{R}^{n-1}} \int_{-\infty}^0 \left(\sum_{j=1}^{k+1} \left| \lambda_j \left(-\frac{1}{j}\right)^{|\alpha_n|} \right|^2 \right) \left(\sum_{j=1}^{k+1} |D^\alpha f(x', -x_n/j)|^2 \right) dx_n \, dx'.$$

Letting $c_1 = \sum_{j=1}^{k+1} |\lambda_j(-1/j)^{|\alpha_n|}|^2$ and making the change of variables $x_n = -s_n j$, we have

$$I \leqslant c_1 \sum_{j=1}^{k+1} j \int_{\mathbb{R}^{n-1}} \int_0^\infty |D^\alpha f(x', s_n)|^2 \, ds_n \, dx'.$$

Letting $c_2 = c_1 \sum_{j=1}^{k+1} j$, we have

$$I \leqslant c_2 \int_{\mathbb{R}^n_+} |D^\alpha f(x)|^2 \, dx.$$

Hence,

$$\int_{\mathbb{R}^n} |D^\alpha Ef(x)|^2 \, dx \leqslant \int_{\mathbb{R}^n_+} |D^\alpha f(x)|^2 \, dx + c_2 \int_{\mathbb{R}^n_+} |D^\alpha f(x)|^2 \, dx$$
$$= (1+c_2) \int_{\mathbb{R}^n_+} |D^\alpha f(x)|^2 \, dx. \quad \blacksquare$$

Definition 2.5. Take $k \in \mathbb{Z}_+$ and let $f \in Y^k(\mathbb{R}^n_+)$. We define

$$\|f\|^2_{\mathbb{R}^n_+} = \sum_{\substack{\alpha \in \mathbb{Z}^n_+ \\ |\alpha| = k}} c_\alpha \, |f|^2_{\alpha, \mathbb{R}^n_+} + \sum_{\substack{\alpha \in \mathbb{Z}^n_+ \\ |\alpha| \leqslant k}} \int_{\mathbb{R}^n_+} |D^\alpha f(x)|^2 \, dx,$$

where the c_α are constants. Let $\| \cdot \|_{\mathbb{R}^n}$ be similarly defined.

Theorem 2.6. *Let $\theta = (0, ..., 0) \in \mathbb{Z}^n_+$. For $f \in \bigcap_{|\alpha| = k} X^\alpha(\mathbb{R}^n_+)$, let $Ef = E_\theta f$ be as defined in Theorem* 2.2. *Then for some constant M independent of f*

$$\|Ef\|_{\mathbb{R}^n} \leqslant M \, \|f\|_{\mathbb{R}^n_+}.$$

Proof. The result follows from Theorem 2.3 and Lemma 2.4. $\blacksquare$

3. SOME PREPARATORY RESULTS

To obtain a more general extension theorem needs a lot of preparation. The informed reader will be able to see that at a very coarse level, our overall *strategy* of proof follows that used in the Sobolev extension theorems. However, at least at the level of detail contained in this section, things are very different. Here we have gathered together many of the technical tools necessary for our more general result. Only one of these seems interesting in its own right, and it has been dignified as Theorem 3.13.

Throughout this section we will refer to the concept of a domain in $\mathbb{R}^n$; that is, an open subset of $\mathbb{R}^n$. Take $k \in \mathbb{Z}_+$ and suppose Ω is a domain in $\mathbb{R}^n$. Let $f \in C^k(\Omega)$. Then,

$$|f|_\Omega = \left(\sum_{|\alpha| = k} c_\alpha \int_\Omega \int_\Omega w(x-y)\, |D^\alpha f(x) - D^\alpha f(y)|^2 \, dx\, dy \right)^{1/2}, \tag{4}$$

where the constants c_α are defined by the algebraic identity

$$\sum_{|\alpha| = k} c_\alpha x^{2\alpha} = |x|^{2k}, \qquad \text{for all} \quad x \in \mathbb{R}^n.$$

As in Section 2, $|\cdot|_\Omega$ is an extended, real number and we denote by $X(\Omega)$ the set of all f restricted to Ω such that $f \in C_0^k(\mathbb{R}^n)$ and $|f|_\Omega < \infty$. The function w is always assumed to be measurable, and often required to satisfy further properties. These properties will always be enough to guarantee that on $X(\Omega)$, $|\cdot|_\Omega$ defines a seminorm with kernel consisting of the polynomials of degree at most k. There is therefore the possibility of constructing a norm from this seminorm. If Ω is a bounded domain and $f \in X(\Omega)$, then

$$\|f\|_\Omega = \left(\sum_{|\alpha| \leqslant k} \int_\Omega |D^\alpha f(x)|^2 \, dx + |f|_\Omega^2 \right)^{1/2} \tag{5}$$

is our preferred choice.

Definition 3.1. Let Ω_1 and Ω_2 be domains in $\mathbb{R}^n$, and Φ a bijection from Ω_1 to Ω_2. We say that Φ is k-smooth if, writing $\Phi(x) = (\phi_1(x_1, ..., x_n), ..., \phi_n(x_1, ..., x_n))$ and $\Phi^{-1}(x) = \Psi(x) = (\psi_1(x_1, ..., x_n), ..., \psi_n(x_1, ..., x_n))$, then the functions $\phi_1, ..., \phi_n$ belong to $C^k(\bar{\Omega}_1)$ and $\psi_1, ..., \psi_n$ belong to $C^k(\bar{\Omega}_2)$. If $k = 0$ then we will refer to Φ as smooth.

Definition 3.2. Let Φ be a bijection from $\mathbb{R}^n$ to $\mathbb{R}^n$. We say Φ is locally k-smooth if Φ is k-smooth on every bounded domain in $\mathbb{R}^n$.

As we have already indicated, assumptions on w are often needed. We gather together all the required hypotheses here:

(W1) $w \in L^1(\mathbb{R}^n \setminus N)$ for any neighbourhood N of the origin;

(W2) $w(y) = \mathcal{O}(|y|^s)$ as $y \to 0$, where $n + s + 2 > 0$;

(W3) $\int_A w > 0$ whenever A has positive measure;

(W4) $w(y) = w(-y)$ for all $y \in \mathbb{R}^n$;

(W5) for every locally $(k+1)$-smooth map ϕ on $\mathbb{R}^n$, and every bounded subset Ω of $\mathbb{R}^n$, there is a $K>0$ such that $w(\phi(x)-\phi(y)) \leqslant Kw(x-y)$, for all $x, y \in \Omega$;

(W6) there exists a constant $M>0$ such that if $x=(x', x_n) \in \mathbb{R}^n$ and $y=(x', y_n) \in \mathbb{R}^n$ with $|x_n| \geqslant |y_n|$, then $w(x) \leqslant Mw(y)$.

LEMMA 3.3. *Let $w\colon \mathbb{R}^n \to \mathbb{R}$ be a measurable function satisfying* (W1)–(W3). *Then the mapping $y \mapsto |y|^2 w(y)$ for $y \in \mathbb{R}^n$ is in $L^1_{loc}(\mathbb{R}^n)$.*

Proof. Choose $\delta>0$ and set $N=\{y \in \mathbb{R}^n : |y|<\delta\}$. Then there exists $A>0$ such that $|w(y)| \leqslant A|y|^s$ for all $y \in N$. Since $w \in L^1(\mathbb{R}^n \backslash N)$, it is clear that the mapping $y \mapsto |y|^2 w(y)$ for $y \in \mathbb{R}^n$ is in $L^1_{loc}(\mathbb{R}^n \backslash N)$. It suffices to show that this same mapping is in $L^1(N)$. For some appropriate constant B,

$$\int_N |y|^2 w(y)\, dy \leqslant A \int_N |y|^{s+2}\, dy \leqslant AB \int_0^\delta r^{n+s+1}\, dr < \infty. \quad \blacksquare$$

LEMMA 3.4. *Let Ω be an open, bounded subset of $\mathbb{R}^n$. Let $w\colon \mathbb{R}^n \to \mathbb{R}$ be a measurable function satisfying* (W1)–(W4). *There exists $A>0$ such that for each $f \in C^1(\Omega)$,*

$$\int_\Omega \int_\Omega w(x-y)\, |f(x)-f(y)|^2\, dx\, dy \leqslant A \sum_{|\alpha|=1} \int_\Omega |D^\alpha f(x)|^2\, dx.$$

Proof. Since $f \in C^1(\Omega)$, Taylor's formula with integral remainder [5, p. 13] allows us to write

$$\begin{aligned} |f(x)-f(y)|^2 &= \left| \int_0^1 \sum_{|\alpha|=1} (y-x)^\alpha D^\alpha f(x+t(y-x))\, dt \right|^2 \\ &\leqslant \left(\int_0^1 1\, dt \right) \left(\int_0^1 \left| \sum_{|\alpha|=1} (y-x)^\alpha D^\alpha f(x+t(y-x)) \right|^2 dt \right) \\ &\leqslant \int_0^1 \left(\sum_{|\alpha|=1} 1 \right) \left(\sum_{|\alpha|=1} |(y-x)^\alpha D^\alpha f(x+t(y-x))|^2 \right) dt. \end{aligned}$$

Now, let χ_Ω be the characteristic function of the set Ω. Extend each $D^\alpha f$ to a function on $\mathbb{R}^n$ by setting it to be zero outside Ω. Two applications of Fubini's theorem plus the change of variables $y=z+x$ gives

$$\begin{aligned}
&\int_\Omega \int_\Omega w(x-y)\, |f(x)-f(y)|^2\, dx\, dy\\
&\quad \leqslant n \sum_{|\alpha|=1} \int_\Omega \int_\Omega w(x-y) \int_0^1 |(y-x)^\alpha D^\alpha f(x+t(y-x))|^2\, dt\, dy\, dx\\
&\quad = n \sum_{|\alpha|=1} \int_0^1 \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} w(x-y)\, \chi_\Omega(x)\, \chi_\Omega(y)\\
&\qquad |(y-x)^\alpha D^\alpha f(x+t(y-x))|^2\, dy\, dx\, dt\\
&\quad = n \sum_{|\alpha|=1} \int_0^1 \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} w(z)\, \chi_\Omega(x)\, \chi_\Omega(x+z)\, |z^\alpha|^2\, |D^\alpha f(x+tz)|^2\, dz\, dx\, dt\\
&\quad = n \sum_{|\alpha|=1} \int_0^1 \int_{\mathbb{R}^n} w(z)\, |z^\alpha|^2 \int_{\Omega \cap (\Omega - z)} |D^\alpha f(x+tz)|^2\, \chi_\Omega(x)\, \chi_\Omega(x+z)\, dx\, dz\, dt.
\end{aligned}$$

Since Ω is bounded, we can find $\delta > 0$ such that if $|z| > \delta$ then $\Omega \cap (\Omega - z)$ is empty. Let $B_\delta = \{y \in \mathbb{R}^n : |y| \leqslant \delta\}$. Then the change of variables $x + tz = v$ gives

$$\begin{aligned}
&\int_\Omega \int_\Omega w(x-y)\, |f(x)-f(y)|^2\, dx\, dy\\
&\quad \leqslant n \sum_{|\alpha|=1} \int_0^1 \int_{B_\delta} w(z)\, |z^\alpha|^2 \int_{\Omega \cap (\Omega - z)} |D^\alpha f(x+tz)|^2\, \chi_\Omega(x)\, \chi_\Omega(x+z)\, dx\, dz\, dt\\
&\quad = n \sum_{|\alpha|=1} \int_0^1 \int_{B_\delta} w(z)\, |z^\alpha|^2 \int_{\mathbb{R}^n} \chi_\Omega(v-tz)\, \chi_\Omega(v+(1-t)\, z)\, |D^\alpha f(v)|^2\, dv\, dz\, dt\\
&\quad \leqslant n \sum_{|\alpha|=1} \int_0^1 \int_{B_\delta} w(z)\, |z^\alpha|^2 \int_{\mathbb{R}^n} |D^\alpha f(v)|^2\, dv\, dz\, dt\\
&\quad \leqslant n \sum_{|\alpha|=1} \int_{B_\delta} w(z)\, |z|^2 \int_\Omega |D^\alpha f(v)|^2\, dv\, dz,
\end{aligned}$$

since $(D^\alpha f)(v) = 0$ for $v \notin \Omega$. Now by Lemma 3.3, there is a constant $A > 0$ independent of f such that

$$\int_\Omega \int_\Omega w(x-y)\, |f(x)-f(y)|^2\, dx\, dy \leqslant An \sum_{|\alpha|=1} \int_\Omega |D^\alpha f(v)|^2\, dv. \quad \blacksquare$$

LEMMA 3.5. *Let U, H, G be measurable subsets of $\mathbb{R}^n$ satisfying the following properties:*

(1) *H is a bounded set and $U \subset H \subset G$;*

(2) *there exists a $\delta > 0$ such that for all $x \in G \setminus H$ and $y \in U$, $|x - y| > \delta$.*

Let $w\colon \mathbb{R}^n \to \mathbb{R}$ *be a measurable function satisfying* (W1). *Then there exists a constant* K *such that for all* $y \in U$,

$$\left| \int_{G \setminus H} w(x-y)\, dx \right| \leqslant K.$$

Proof. Define $f: U \to \mathbb{R}$ by $f(y) = \int_{G\setminus H} w(x-y)\, dx$ for $y \in U$. Making the change of variables $x = s + y$ gives

$$f(y) = \int_{T_y} w(s)\, ds,$$

where $T_y = G \setminus H - y$. Take $s \in T_y$. Then $s = x - y$ for some $x \in G \setminus H$ and so by Condition (2), $|x-y| > \delta$. Now take $N = \{s \in \mathbb{R}^n : |s| < \delta\}$. Then $T_y \subset \mathbb{R}^n \setminus N$ and

$$|f(y)| = \left| \int_{T_y} w(s)\, ds \right| \leqslant \int_{T_y} |w(s)|\, ds \leqslant \int_{\mathbb{R}^n \setminus N} |w(s)|\, ds.$$

Setting $K = \int_{\mathbb{R}^n \setminus N} |w(s)|\, ds$ gives the result. ▮

LEMMA 3.6. *Let* H *be a bounded subset of* $\mathbb{R}^n$. *Let* U *be a subset of* H *such that* $H \setminus U$ *has positive measure. Let* $w\colon \mathbb{R}^n \to \mathbb{R}$ *be a measurable function satisfying* (W1) *and* (W3). *Then there is a number* $K > 0$ *such that*

$$\int_{H \setminus U} w(x-y)\, dx \geqslant K, \qquad \textit{for all} \quad y \in U.$$

Proof. Define f from $\mathbb{R}^n$ to the extended reals by

$$f(y) = \int_{H \setminus U} w(x-y)\, dx = \int_{T_y} w(s)\, ds,$$

where $T_y = H \setminus U - y$ and $y \in \mathbb{R}^n$. Because T_y has positive measure, $f(y) > 0$ for all $y \in \mathbb{R}^n$. We claim f is a lower semicontinuous function on $\mathbb{R}^n$. That is, the set $Y_\alpha = \{y \in \mathbb{R}^n : f(y) > \alpha\}$ is open for each $\alpha \in \mathbb{R}$. Clearly if $\alpha \leqslant 0$ then Y_α is the whole of $\mathbb{R}^n$ and so is open. Thus we fix $\alpha > 0$. We will show that the set $Y_\alpha^c = \{y \in \mathbb{R} : f(y) \leqslant \alpha\}$ is closed. Let $\{v_j\}_{j=0}^{\infty}$ be a sequence in Y_α^c. Then

$$f(v_j) = \int_{T_{v_j}} w(x)\, dx \leqslant \alpha, \qquad \text{for all} \quad j = 0, 1, \ldots.$$

For convenience we shall write T_j for T_{v_j}. Suppose that $\lim_{j\to\infty} v_j = v$. We wish to show that $v \in Y_\alpha^c$. Let N be any neighbourhood of the origin. We define $A = T_v \cap N$ and $A_j = T_j \cap N$. Since $w \in L^1(\mathbb{R}^n \setminus N)$ we have

$$\int_{T_v \setminus A} w(x)\, dx = \lim_{j\to\infty} \int_{T_j \setminus A_j} w(x)\, dx \leqslant \alpha. \tag{6}$$

Let $B(0, 1/m) = \{x \in \mathbb{R}^n : |x| < 1/m\}$ and define $L_m = T_v \cap B(0, 1/m)$. Let χ_m be the characteristic function of L_m. Consider the sequence $\{w_k\}_{k=1}^\infty$ defined by $w_k = (1-\chi_k)\, w$. Now, for all $x \in \mathbb{R}^n$,

(i) $0 \leqslant w_0(x) \leqslant w_1(x) \leqslant \ldots$

(ii) $\lim_{k\to\infty} w_k(x) = w(x)$.

Note that in order to ensure Condition (ii) for $x = 0$, we need to define $w(0) = 0$. Now, the Lebesgue Monotone Convergence Theorem and Eq. (6) give

$$\int_{T_v} w(x)\, dx = \lim_{k\to\infty} \int_{T_v} (1-\chi_k)(x)\, w(x)\, dx = \lim_{k\to\infty} \int_{T_v \setminus L_k} w(x)\, dx \leqslant \alpha.$$

Therefore, $v \in Y_\alpha^c$ and Y_α^c is closed. Hence, f is lower semicontinuous. Since $U \subset H$ and H is bounded, U lies in some closed ball, centred on the origin. Now f attains its (positive) infimum on this ball, and so the required conclusion follows. ∎

The following result, which for a long time we referred to as the "secret lemma," seems to us to be absolutely crucial in all extensions theorems of this nature. It tells us that under appropriate circumstances, contributions of integrals over sets of the form $G \times (G \setminus H)$ can be in some sense disregarded.

LEMMA 3.7. *Let $U \subset H \subset G$ be measurable subsets of $\mathbb{R}^n$, with H bounded. Suppose that for some $\delta > 0$, $|x-y| > \delta$ for all $x \in G \setminus H$ and $y \in U$. Suppose w: $\mathbb{R}^n \to \mathbb{R}$ is a measurable function satisfying* (W1), (W3) *and* (W4). *Let X consist of all functions $f \in C(G)$ for which the mapping F: $G \times G \to \mathbb{R}$ given by $F(x, y) = w(x-y)\, |f(x) - f(y)|^2$ for $x, y \in \mathbb{R}^n$ is in $L^1(G \times G)$. There is a number K such that*

$$\int_G \int_G F(x, y)\, dx\, dy \leqslant K \int_H \int_H F(x, y)\, dx\, dy,$$

for all $f \in X$ with support in U.

Proof. Because $F \in L^1(G \times G)$, and f is supported on U we can write

$$\int_G \int_G F(x, y)\, dx\, dy = 2 \int_U \int_{G \setminus U} F(x, y)\, dx\, dy + \int_U \int_U F(x, y)\, dx\, dy$$

$$= 2 \int_U \int_{G \setminus H} F(x, y)\, dx\, dy + 2 \int_U \int_{H \setminus U} F(x, y)\, dx\, dy$$

$$+ \int_U \int_U F(x, y)\, dx\, dy$$

$$= 2 \int_U \int_{G \setminus H} F(x, y)\, dx\, dy + \int_H \int_H F(x, y)\, dx\, dy.$$

Now, again using the facts that $F \in L^1(G \times G)$, and f is supported in U,

$$\int_U \int_{G \setminus H} F(x, y)\, dx\, dy = \int_U |f(y)|^2 \int_{G \setminus H} w(x-y)\, dx\, dy.$$

Lemmas 3.5 and 3.6 show that there exists constants $K_1, K_2 > 0$ such that

$$\int_{G \setminus H} w(x-y)\, dx \leqslant K_1 \leqslant \frac{K_1}{K_2} \int_{H \setminus U} w(x-y)\, dx.$$

Since f is supported on U, we conclude that

$$\int_U \int_{G \setminus H} F(x, y)\, dx\, dy \leqslant \frac{K_1}{K_2} \int_U |f(y)|^2 \int_{H \setminus U} w(x-y)\, dx\, dy$$

$$= \frac{K_1}{K_2} \int_U \int_{H \setminus U} F(x, y)\, dx\, dy.$$

Finally,

$$\int_G \int_G F(x, y)\, dx\, dy \leqslant \frac{2K_1}{K_2} \int_U \int_{H \setminus U} F(x, y)\, dx\, dy + \int_H \int_H F(x, y)\, dx\, dy$$

$$\leqslant \left(\frac{K_1}{K_2} + 1 \right) \int_H \int_H F(x, y)\, dx\, dy. \quad \blacksquare$$

Our extension theorem for more general domains depends on an understanding of k-smooth mappings. The following four Lemmas, culminating in Theorem 3.13, establish the necessary results in this area.

LEMMA 3.8. *Let Ω_1, Ω_2 be domains in $\mathbb{R}^n$ and ϕ a k-smooth bijection from Ω_1 to Ω_2. For each $f \in C^k(\Omega_2)$ and $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \leqslant k$,*

$$D^\alpha (f \circ \phi) = \sum_{0 \leqslant |\beta| \leqslant |\alpha|} P_{\alpha\beta} [(D^\beta f) \circ \phi], \tag{7}$$

where each $P_{\alpha\beta}$ is a polynomial of degree at most $|\beta|$ in derivatives of the components of ϕ of orders at most $|\alpha|$.

Proof. The proof is by induction on $|\alpha|$. If $\alpha = 0$, then the result holds with $P_{00} = 1$. Now assume Equation (7) holds for all $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| < m \leqslant k$. Take $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| = m$. Then $\alpha = \beta + \gamma$ where $|\beta| < m$ and $|\gamma| = 1$. Now employing the induction hypothesis,

$$\begin{aligned} D^\alpha(f \circ \phi) &= D^\gamma D^\beta(f \circ \phi) \\ &= D^\gamma\left(\sum_{0 \leqslant |\nu| \leqslant m-1} P_{\beta\nu}[(D^\nu f) \circ \phi]\right) \\ &= \sum_{0 \leqslant |\nu| \leqslant m-1} ((D^\gamma P_{\beta\nu})[(D^\nu f) \circ \phi] + P_{\beta\nu} D^\gamma[(D^\nu f) \circ \phi]). \end{aligned}$$

The induction hypothesis can now be employed again on part of the second term in the parentheses above giving

$$\begin{aligned} D^\gamma[(D^\nu f) \circ \phi] &= \sum_{0 \leqslant |\mu| \leqslant 1} P_{\gamma\mu}[(D^{\mu+\nu} f) \circ \phi] \\ &= P_{\gamma 0}[(D^\nu f) \circ \phi] + \sum_{|\mu| = 1} P_{\gamma\mu}[(D^{\mu+\nu} f) \circ \phi]. \end{aligned}$$

Thus,

$$\begin{aligned} D^\alpha(f \circ \phi) &= \sum_{0 \leqslant |\nu| \leqslant m-1} (D^\gamma P_{\beta\nu} + P_{\beta\nu} P_{\gamma 0})[(D^\nu f) \circ \phi] \\ &\quad + \sum_{0 \leqslant |\nu| \leqslant m-1} P_{\beta\nu} \sum_{|\mu| = 1} P_{\gamma\mu}[(D^{\mu+\nu} f) \circ \phi] \\ &= \sum_{0 \leqslant |\nu| \leqslant m-1} (D^\gamma P_{\beta\nu} + P_{\beta\nu} P_{\gamma 0})[(D^\nu f) \circ \phi] \\ &\quad + \sum_{1 \leqslant |\nu| \leqslant m} \left(\sum_{\substack{\mu+\delta=\nu \\ |\mu|=1 \\ \delta \geqslant 0}} P_{\beta\delta} P_{\gamma\mu}\right)[(D^\nu f) \circ \phi]. \end{aligned}$$

We can therefore write

$$D^\alpha(f \circ \phi) = \sum_{0 \leqslant |\nu| \leqslant m} P_{\alpha\nu}[(D^\nu f) \circ \phi],$$

where

$$P_{\alpha\nu} = \begin{cases} D^{\gamma}P_{\beta 0} + P_{\beta 0}P_{\gamma 0}, & \nu = 0 \\ D^{\gamma}P_{\beta\nu} + P_{\beta\nu}P_{\gamma 0} + \sum\limits_{\substack{\mu+\delta=\nu \\ |\mu|=1 \\ \delta\geqslant 0}} P_{\beta\delta}P_{\gamma\mu}, & 1 \leqslant |\nu| \leqslant m-1 \\ \sum\limits_{\substack{\mu+\delta=\nu \\ |\mu|=1 \\ \delta\geqslant 0}} P_{\beta\delta}P_{\gamma\mu}, & |\nu| = m. \end{cases}$$

The result now follows by induction. ∎

LEMMA 3.9. *Let ϕ be a k-smooth bijection between bounded domains Ω_1 and Ω_2 in $\mathbb{R}^n$. There exists a constant K such that for all $\alpha \in \mathbb{Z}^n_+$ with $|\alpha| \leqslant k$ and for all $f \in C^k(\Omega_2)$,*

$$\int_{\Omega_1} |D^{\alpha}(f \circ \phi)(x)|^2\, dx \leqslant K \max_{|\beta| \leqslant |\alpha|} \int_{\Omega_2} |(D^{\beta} f)(x)|^2\, dx.$$

Proof. Take $f \in C^k(\Omega_2)$ and $\alpha \in \mathbb{Z}^n_+$ with $|\alpha| \leqslant k$. Then, using Lemma 3.8 and the Cauchy–Schwarz inequality,

$$\begin{aligned} &\int_{\Omega_1} |D^{\alpha}(f \circ \phi)(x)|^2\, dx \\ &\quad = \int_{\Omega_1} \left| \sum_{|\beta| \leqslant |\alpha|} P_{\alpha\beta}(x)[(D^{\beta} f) \circ \phi](x) \right|^2 dx \\ &\quad \leqslant \int_{\Omega_1} \left(\sum_{|\beta| \leqslant |\alpha|} 1 \right) \left(\sum_{|\beta| \leqslant |\alpha|} |P_{\alpha\beta}(x)|^2\, |[(D^{\beta} f) \circ \phi](x)|^2 \right) dx \\ &\quad \leqslant \left(\sum_{|\beta| \leqslant |\alpha|} 1 \right)^2 \max_{|\beta| \leqslant |\alpha|} \int_{\Omega_1} |P_{\alpha\beta}(x)|^2\, |[(D^{\beta} f) \circ \phi](x)|^2\, dx \\ &\quad \leqslant \left(\sum_{|\beta| \leqslant |\alpha|} 1 \right)^2 \max_{|\beta| \leqslant |\alpha|} \left(\max_{x \in \Omega_1} |P_{\alpha\beta}(x)|^2 \int_{\Omega_1} |[(D^{\beta} f) \circ \phi](x)|^2\, dx \right). \end{aligned}$$

Now suppose the maximum above over $|\beta| \leqslant |\alpha|$ occurs at $\beta = \beta_0$. Since Ω_1 is a bounded domain, we can assume that there is a number K_1 such that

$$\left(\sum_{|\beta| \leqslant |\alpha|} 1 \right)^2 \max_{x \in \Omega_1} |P_{\alpha\beta_0}(x)|^2 \leqslant K_1.$$

Making the change of variables $x = \phi^{-1}(y)$, we obtain

$$\int_{\Omega_1} |D^\alpha(f \circ \phi)(x)|^2 \, dx \leqslant K_1 \int_{\Omega_1} |(D^{\beta_0} f)(\phi(x))|^2 \, dx$$

$$\leqslant K_1 \int_{\Omega_2} |(D^{\beta_0} f)(y)|^2 \, |J_{\phi^{-1}}(y)| \, dy,$$

where $J_{\phi^{-1}}$ is the corresponding Jacobian. Since Ω_2 is bounded, this Jacobian is bounded on Ω_2, and so there is a number K_2, such that

$$\int_{\Omega_1} |D^\alpha(f \circ \phi)(x)|^2 \, dx \leqslant K_1 K_2 \int_{\Omega_2} |(D^{\beta_0} f)(x)|^2 \, dx$$

as required. ∎

Lemma 3.10. *Let ϕ be a $(k+1)$-smooth bijection between bounded domains Ω_1 and Ω_2 in $\mathbb{R}^n$. Let $\alpha, \beta \in \mathbb{Z}_+^n$ with $|\alpha|, |\beta| \leqslant k$. Let $P_{\alpha\beta}$ be as in Lemma* 3.8. *Let w be a measurable function satisfying* (W1)–(W3). *Then there exists a constant K such that*

$$\int_{\Omega_1} w(x-y) \, |P_{\alpha\beta}(x) - P_{\alpha\beta}(y)|^2 \, dx \leqslant K,$$

for all $y \in \Omega_1$.

Proof. Recall from Lemma 3.8, that $P_{\alpha\beta}$ is a polynomial of degree at most $|\beta|$ in derivatives of the components of ϕ of orders at most $|\alpha|$. Let $\phi(x) = (\phi_1(x_1, ..., x_n), ..., \phi_n(x_1, ..., x_n))$. Because ϕ is $(k+1)$-smooth, the functions $\phi_1, ..., \phi_n$ are in $C^{k+1}(\bar{\Omega}_1)$. Hence, we can find a constant K_1 such that for all $1 \leqslant i \leqslant n$,

$$|(D^\gamma \phi_i)(x) - (D^\gamma \phi_i)(y)| \leqslant K_1 \, |x-y|,$$

for all $x, y \in \Omega_1$ and for all $\gamma \in \mathbb{Z}_+^n$ with $|\gamma| \leqslant k$. Consequently, we can find a constant K_2 such that $|P_{\alpha\beta}(x) - P_{\alpha\beta}(y)| \leqslant K_2 \, |x-y|$ for all $x, y \in \Omega_1$ and for all $\alpha, \beta \in \mathbb{Z}_+^n$ with $|\alpha|, |\beta| \leqslant k$. Hence,

$$\int_{\Omega_1} w(x-y) \, |P_{\alpha\beta}(x) - P_{\alpha\beta}(y)|^2 \, dx \leqslant K_2^2 \int_{\Omega_1} |x-y|^2 \, w(x-y) \, dx.$$

Using the change of variables $x - y = s$ we have

$$\int_{\Omega_1} w(x-y) \, |P_{\alpha\beta}(x) - P_{\alpha\beta}(y)|^2 \, dx \leqslant K_2^2 \int_{\Omega_1 - y} |s|^2 \, w(s) \, ds.$$

Lemma 3.3 establishes the existence of a constant $K_3(y) > 0$ such that

$$\int_{\Omega_1} w(x-y)\, |P_{\alpha\beta}(x) - P_{\alpha\beta}(y)|^2\, dx \leqslant K_2^2 K_3(y).$$

Again by Lemma 3.3, the map $s \mapsto |s|^2\, w(s)$ is in $L^1_{loc}(\mathbb{R}^n)$. Therefore, the function $y \mapsto \int_{\Omega_1 - y} |s|^2\, w(s)\, ds$ is continuous. Since Ω_1 is bounded, it follows that $\sup_{y \in \Omega_1} K_3(y) < \infty$. Thus the required result is obtained by taking

$$K = K_2^2 \sup_{y \in \Omega_1} K_3(y). \quad ∎$$

In the following result we will make use of the following simple inequality. For all $a, b \in \mathbb{R}^l$,

$$|a+b|^2 \leqslant |a|^2 + 2\,|a|\,|b| + |b|^2 \leqslant 3(|a|^2 + |b|^2). \tag{8}$$

LEMMA 3.11. *Let ϕ be a $(k+1)$-smooth bijection between bounded domains Ω_1 and Ω_2 in $\mathbb{R}^n$. Let w be a measurable function satisfying* (W1)–(W3) *and* (W5). *There exists a constant K such that for all $f \in C^k(\Omega_2)$ and all $\alpha \in \mathbb{Z}^n_+$ with $|\alpha| \leqslant k$,*

$$\begin{aligned}
&\int_{\Omega_1} \int_{\Omega_1} w(x-y)\, |D^\alpha(f \circ \phi)(x) - D^\alpha(f \circ \phi)(y)|^2\, dx\, dy \\
&\qquad \leqslant K \max_{|\beta| \leqslant |\alpha|} \int_{\Omega_2} \int_{\Omega_2} w(x-y)\, |(D^\beta f)(x) - (D^\beta f)(y)|^2\, dx\, dy \\
&\qquad\qquad + K \max_{|\beta| \leqslant |\alpha|} \int_{\Omega_2} |(D^\beta f)(x)|^2\, dx.
\end{aligned}$$

Proof. Take $f \in C^k(\Omega_2)$ and $\alpha \in \mathbb{Z}^n_+$ such that $|\alpha| \leqslant k$. Observe first that by Lemma 3.8, the Cauchy–Schwarz inequality and the remark preceding this lemma,

$$\begin{aligned}
&|D^\alpha(f \circ \phi)(x) - D^\alpha(f \circ \phi)(y)|^2 \\
&\quad = \left| \sum_{|\beta| \leqslant |\alpha|} (P_{\alpha\beta}(x)(D^\beta f \circ \phi)(x) - P_{\alpha\beta}(y)(D^\beta f \circ \phi)(y)) \right|^2 \\
&\quad \leqslant \left(\sum_{|\beta| \leqslant |\alpha|} 1 \right) \left(\sum_{|\beta| \leqslant |\alpha|} |P_{\alpha\beta}(x)(D^\beta f \circ \phi)(x) - P_{\alpha\beta}(y)(D^\beta f \circ \phi)(y)|^2 \right) \\
&\quad \leqslant 3 \left(\sum_{|\beta| \leqslant |\alpha|} 1 \right) \left(\sum_{|\beta| \leqslant |\alpha|} |P_{\alpha\beta}(x)|^2\, |(D^\beta f \circ \phi)(x) - (D^\beta f \circ \phi)(y)|^2 \right. \\
&\qquad\qquad \left. + \sum_{|\beta| \leqslant |\alpha|} |(D^\beta f \circ \phi)(y)|^2\, |P_{\alpha\beta}(x) - P_{\alpha\beta}(y)|^2 \right).
\end{aligned}$$

Put $K_1 = 3 \sum_{|\beta| \leqslant |\alpha|} 1$. Then

$$\int_{\Omega_1} \int_{\Omega_1} w(x-y)\, |D^\alpha(f \circ \phi)(x) - D^\alpha(f \circ \phi)(y)|^2\, dx\, dy$$

$$\leqslant K_1 \sum_{|\beta| \leqslant |\alpha|} \int_{\Omega_1} \int_{\Omega_1} w(x-y)\, |P_{\alpha\beta}(x)|^2\, |(D^\beta f \circ \phi)(x) - (D^\beta f \circ \phi)(y)|^2\, dx\, dy$$

$$+ K_1 \sum_{|\beta| \leqslant |\alpha|} \int_{\Omega_1} |(D^\beta f \circ \phi)(y)|^2 \int_{\Omega_1} w(x-y)\, |P_{\alpha\beta}(x) - P_{\alpha\beta}(y)|^2\, dx\, dy.$$

We examine each of the above integrals in turn. First, since Ω_1 is bounded we can assume that $|P_{\alpha\beta}(x)|^2 \leqslant K_2$ for all $|\beta| \leqslant |\alpha|$ and for all $x \in \Omega_1$. Thus, making the changes of variables $x = \phi^{-1}(s)$ and $y = \phi^{-1}(t)$,

$$\int_{\Omega_1} \int_{\Omega_1} w(x-y)\, |P_{\alpha\beta}(x)|^2\, |(D^\beta f \circ \phi)(x) - (D^\beta f \circ \phi)(y)|^2\, dx\, dy$$

$$\leqslant K_2 \int_{\Omega_1} \int_{\Omega_1} w(x-y)\, |(D^\beta f \circ \phi)(x) - (D^\beta f \circ \phi)(y)|^2\, dx\, dy$$

$$\leqslant K_2 \int_{\Omega_2} \int_{\Omega_2} w(\phi^{-1}(s) - \phi^{-1}(t))\, |(D^\beta f)(s) - (D^\beta f)(t)|^2$$

$$|J_{\phi^{-1}}(s)\, J_{\phi^{-1}}(t)|\, ds\, dt.$$

Using hypothesis (W5) and the fact that $|J_{\phi^{-1}}|$ is bounded on the domain Ω_2, we infer the existence of a constant K_3 such that

$$\int_{\Omega_1} \int_{\Omega_1} w(x-y)\, |P_{\alpha\beta}(x)|^2\, |(D^\beta f \circ \phi)(x) - (D^\beta f \circ \phi)(y)|^2\, dx\, dy$$

$$\leqslant K_3 \int_{\Omega_2} \int_{\Omega_2} w(s-t)\, |(D^\beta f)(s) - (D^\beta f)(t)|^2\, ds\, dt.$$

Moreover, by Lemma 3.10 there is a constant $K_4 \geqslant K_3$ such that

$$\int_{\Omega_1} |(D^\beta f \circ \phi)(y)|^2 \int_{\Omega_1} w(x-y)\, |P_{\alpha\beta}(x) - P_{\alpha\beta}(y)|^2\, dx\, dy$$

$$\leqslant K_4 \int_{\Omega_1} |[D^\beta f \circ \phi](x)|^2\, dx$$

$$\leqslant K_5 \max_{|\beta| \leqslant |\alpha|} \int_{\Omega_2} |(D^\beta f)(x)|^2\, dx.$$

The last inequality is a consequence of Lemma 3.9. Assuming (with no loss of generality) that $K_5 \geqslant K_3$,

$$\int_{\Omega_1}\int_{\Omega_1} w(x-y)\,|D^\alpha(f\circ\phi)(x)-D^\alpha(f\circ\phi)(y)|^2\,dx\,dy$$

$$\leqslant K_1K_3\sum_{|\beta|\leqslant|\alpha|}\int_{\Omega_2}\int_{\Omega_2} w(s-t)\,|(D^\beta f)(s)-(D^\beta f)(t)|^2\,ds\,dt$$

$$+K_1K_5\sum_{|\beta|\leqslant|\alpha|}\max_{|\beta|\leqslant|\alpha|}\int_{\Omega_2}|(D^\beta f)(x)|^2\,dx$$

$$\leqslant K_1K_5\left(\sum_{|\beta|\leqslant|\alpha|}1\right)\left(\max_{|\beta|\leqslant|\alpha|}\int_{\Omega_2}\int_{\Omega_2} w(s-t)\,|(D^\beta f)(s)-(D^\beta f)(t)|^2\,ds\,dt\right.$$

$$\left.+\max_{|\beta|\leqslant|\alpha|}\int_{\Omega_2}|(D^\beta f)(x)|^2\,dx\right).$$

Taking $K=K_1K_5\sum_{|\beta|\leqslant|\alpha|}1$ completes the proof. ∎

We now come to one of the central results of this section. To understand it, we introduce a further piece of notation.

Definition 3.12. Let Ω be a domain in $\mathbb{R}^n$ and $f\in C^k(\Omega)$. Let $w\colon\mathbb{R}^n\to\mathbb{R}$ be a measurable function satisfying (W1)–(W6). Set

$$\|f\|_\Omega=\left(\sum_{|\alpha|\leqslant k}\int_\Omega|(D^\alpha f)(x)|^2\,dx\right.$$

$$\left.+\sum_{|\alpha|=k}c_\alpha\int_\Omega\int_\Omega w(x-y)\,|(D^\alpha f)(x)-(D^\alpha f)(y)|^2\,dx\,dy\right)^{1/2}.$$

The symbol $\|\cdot\|_\Omega$ is not used inadvisedly here, since it defines a norm on $X(\Omega)$ as introduced at the start of this section.

Theorem 3.13. *Let ϕ be a $(k+1)$-smooth bijection from a bounded domain Ω_1 into $\mathbb{R}^n$. Let $w\colon\mathbb{R}^n\to\mathbb{R}$ be a measurable function satisfying* (W1)–(W6). *Then there is a number K such that*

$$\|f\circ\phi\|_{\Omega_1}\leqslant K\,\|f\|_{\phi(\Omega_1)},\qquad \textit{for all}\quad f\in X(\phi(\Omega_1)).$$

Proof. Set $\Omega_2=\phi(\Omega_1)$. From Lemmas 3.9 and 3.11 we infer the existence of a constant $K_1\geqslant 0$ such that

$$\|f \circ \phi\|_{\Omega_1}^2 = \sum_{|\alpha| \leqslant k} \int_{\Omega_1} |D^\alpha(f \circ \phi)(x)|^2 \, dx$$

$$+ \sum_{|\alpha| = k} c_\alpha \int_{\Omega_1} \int_{\Omega_1} w(x-y) \, |D^\alpha(f \circ \phi)(x) - D^\alpha(f \circ \phi)(y)|^2 \, dx \, dy$$

$$\leqslant K_1 \max_{|\beta| \leqslant k} \int_{\Omega_2} \int_{\Omega_2} w(x-y) \, |D^\beta f(x) - D^\beta f(y)|^2 \, dx \, dy$$

$$+ K_1 \max_{|\beta| \leqslant k} \int_{\Omega_2} |D^\beta f(x)|^2 \, dx.$$

From Lemma 3.4 we infer the existence of a constant $K_2 > 0$ such that

$$\|f \circ \phi\|_{\Omega_1}^2 \leqslant K_1 \Bigg(K_2 \sum_{|\gamma| \leqslant k} \int_{\Omega_2} |D^\gamma f(x)|^2 \, dx$$

$$+ \sum_{|\gamma| = k} c_\alpha \int_{\Omega_2} \int_{\Omega_2} |w(x-y) \,| \, D^\gamma f(x) - D^\gamma f(y)|^2 \, dx \, dy \Bigg)$$

$$+ K_1 \max_{|\beta| \leqslant k} \int_{\Omega_2} |D^\beta f(x)|^2 \, dx$$

$$\leqslant K_1(K_2 + 2) \, \|f\|_{\Omega_2},$$

as required. ∎

LEMMA 3.14. *Let* $u \in C_0^\infty(\mathbb{R}^n)$ *and let* Ω *be a bounded domain. Let* $w\colon \mathbb{R}^n \to \mathbb{R}$ *satisfy* (W1)–(W4). *There exists a constant* C *such that for all* $\gamma \in \mathbb{Z}_+^n$ *with* $|\gamma| = k$,

$$\int_\Omega \int_\Omega w(x-y) \, |D^\gamma(uf)(x) - D^\gamma(uf)(y)|^2 \, dx \, dy \leqslant C \, \|f\|_\Omega^2,$$

for all $f \in X(\Omega)$.

Proof. Let

$$I_1 = \int_\Omega \int_\Omega w(x-y) \, |D^\gamma(uf)(x) - D^\gamma(uf)(y)|^2 \, dx \, dy.$$

The Leibniz formula allows us to write

$$D^\gamma(uf) = \sum_{\substack{\beta \in \mathbb{Z}_+^n \\ |\beta| \leqslant |\gamma|}} C_{\gamma\beta} (D^{\gamma-\beta} u)(D^\beta f),$$

where the $C_{\gamma\beta}$ are suitable numbers. Using this and the Cauchy–Schwarz inequality gives

$$I_1 = \int_\Omega \int_\Omega w(x-y) \left| \sum_{|\beta| \leqslant |\gamma|} C_{\gamma\beta} \{(D^{\gamma-\beta}u)(x)(D^\beta f)(x) - (D^{\gamma-\beta}u)(y)(D^\beta f)(y)\} \right|^2 dx\, dy$$

$$\leqslant \left(\sum_{|\beta| \leqslant |\gamma|} |C_{\gamma\beta}|^2 \right) \int_\Omega \int_\Omega w(x-y) \left(\sum_{|\beta| \leqslant |\gamma|} |(D^{\gamma-\beta}u)(x)(D^\beta f)(x) - (D^{\gamma-\beta}u)(y)(D^\beta f)(y)|^2 \right) dx\, dy.$$

Now set $c_1 = \sum_{|\beta| \leqslant |\gamma|} |C_{\gamma\beta}|^2$. Then using inequality (8) we obtain

$$I_1 \leqslant 3c_1 \sum_{|\beta| \leqslant |\gamma|} \int_\Omega \int_\Omega w(x-y)\, |D^{\gamma-\beta}u(x)|^2\, |D^\beta f(x) - D^\beta f(y)|^2\, dx\, dy$$
$$+ 3c_1 \sum_{|\beta| \leqslant |\gamma|} \int_\Omega \int_\Omega w(x-y)\, |D^\beta f(y)|^2\, |D^{\gamma-\beta}u(x) - D^{\gamma-\beta}u(y)|^2\, dx\, dy.$$

Now set

$$c_2 = \max_{|\beta| \leqslant |\gamma|} \sup_{x \in \Omega} |D^{\gamma-\beta}u(x)|^2.$$

Lemma 3.4 shows that there is a constant c_3 such that

$$I_1 \leqslant 3c_1c_2 \sum_{|\beta| \leqslant |\gamma|} \int_\Omega \int_\Omega w(x-y)\, |D^\beta f(x) - D^\beta f(y)|^2\, dx\, dy$$
$$+ 3c_1 \sum_{|\beta| \leqslant |\gamma|} \int_\Omega \int_\Omega w(x-y)\, |D^\beta f(y)|^2\, |D^{\gamma-\beta}u(x) - D^{\gamma-\beta}u(y)|^2\, dx\, dy$$
$$\leqslant 3c_1c_2 \sum_{|\beta| = k} \int_\Omega \int_\Omega w(x-y)\, |D^\beta f(x) - D^\beta f(y)|^2\, dx\, dy$$
$$+ 3c_1c_2c_3 \sum_{1 \leqslant |\beta| \leqslant k} \int_\Omega |D^\beta f(y)|^2\, dy$$
$$+ 3c_1 \sum_{|\beta| \leqslant |\gamma|} \int_\Omega |D^\beta f(y)|^2 \int_\Omega w(x-y)\, |D^{\gamma-\beta}u(x) - D^{\gamma-\beta}u(y)|^2\, dx\, dy.$$

If we can now show that for each $y \in \Omega$ and every $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \leqslant k$,

$$I_2(y) := \int_\Omega w(x-y)\, |D^\alpha u(x) - D^\alpha u(y)|^2\, dx$$

is bounded by a constant c_4 dependent only on u and α, then we will obtain

$$I_1 \leqslant 3c_1c_2 \sum_{|\beta|=k} \int_\Omega \int_\Omega w(x-y)\, |D^\beta f(x) - D^\beta f(y)|^2\, dx\, dy$$

$$+ 3c_1c_2c_3 \sum_{1 \leqslant |\beta| \leqslant k} \int_\Omega |D^\beta f(y)|^2\, dy$$

$$+ 3c_1c_4 \sum_{|\beta| \leqslant |\gamma|} \int_\Omega |D^\beta f(y)|^2\, dy.$$

This completes the proof. For the boundedness of I_2, we note that since $u \in C_0^\infty(\mathbb{R}^n)$, there exists a constant $c_5(\alpha)$ dependent on α, such that

$$|D^\alpha u(x) - D^\alpha u(y)| \leqslant c_5(\alpha)\, |x-y|,$$

for all $x, y \in \mathbb{R}^n$. Using the change of variables $x - y = s$, we obtain

$$I_2(y) \leqslant c_5(\alpha) \int_\Omega w(x-y)\, |x-y|^2\, dx$$

$$= c_5(\alpha) \int_{\Omega - y} w(s)\, |s|^2\, ds.$$

Lemma 3.3 now establishes the boundedness of I_2 on Ω. ∎

LEMMA 3.15. *Let Ω be a bounded, open subset of $\mathbb{R}^n$. Let $w: \mathbb{R}^n \to \mathbb{R}$ be a measurable function satisfying* (W1)–(W4). *Let $u \in C_0^\infty(\mathbb{R}^n)$. Then there is a number $C > 0$ such that $\|uf\|_\Omega \leqslant C\, \|f\|_\Omega$ for all $f \in X(\Omega)$.*

Proof. Let $f \in X(\Omega)$. An application of Lemma 3.14 shows that

$$\|uf\|_\Omega^2 = \sum_{|\alpha|=k} c_\alpha \int_\Omega \int_\Omega w(x-y)\, |D^\alpha(uf)(x) - D^\alpha(uf)(y)|^2\, dx\, dy$$

$$+ \sum_{|\alpha| \leqslant k} \int_\Omega |D^\alpha(uf)(x)|^2\, dx$$

$$\leqslant \sum_{|\alpha|=k} c_\alpha c_1\, \|f\|_\Omega^2 + \sum_{|\alpha| \leqslant k} \int_\Omega |D^\alpha(uf)(x)|^2\, dx, \tag{9}$$

for some c_1 independent of f. The Leibniz formula guarantees the existence of constants $c_{\alpha\beta}$ such that

$$D^\alpha(uf) = \sum_{|\beta| \leqslant |\alpha|} c_{\alpha\beta} (D^{\alpha-\beta} u)(D^\beta f).$$

Hence, for any $\alpha \in \mathbb{Z}^n_+$ with $|\alpha| = k$, an application of the Cauchy–Schwarz inequality gives

$$\int_\Omega |D^\alpha(uf)(x)|^2\,dx = \int_\Omega \left| \sum_{|\beta| \leqslant |\alpha|} c_{\alpha\beta} D^{\alpha-\beta} u(x)\, D^\beta f(x) \right|^2 dx$$

$$\leqslant \left(\sum_{|\beta| \leqslant |\alpha|} |c_{\alpha\beta}|^2 \right) \int_\Omega \sum_{|\beta| \leqslant |\alpha|} |D^{\alpha-\beta} u(x)\, D^\beta f(x)|^2\,dx.$$

Setting

$$c_2 = \sum_{|\beta| \leqslant |\alpha|} |c_{\alpha\beta}|^2 \max_{|\beta| \leqslant |\alpha|} \sup_{x \in \Omega} |D^{\alpha-\beta} u(x)|^2$$

gives

$$\int_\Omega |D^\alpha(uf)(x)|^2\,dx \leqslant c_2 \sum_{|\beta| \leqslant |\alpha|} \int_\Omega |D^\beta f(x)|^2\,dx \leqslant c_2\, \|f\|^2_\Omega.$$

Substituting this result back in (9) gives

$$\|uf\|^2_\Omega \leqslant \sum_{|\alpha| = k} c_\alpha c_1\, \|f\|^2_\Omega + \sum_{|\alpha| \leqslant k} c_2\, \|f\|^2_\Omega,$$

which is the required result providing we take

$$C \geqslant \sqrt{\sum_{|\alpha| = k} c_\alpha c_1 + \sum_{|\alpha| \leqslant k} c_2}. \quad \blacksquare$$

4. EXTENSION THEOREMS FOR MORE GENERAL DOMAINS

This section contains the main achievement of our work-extension theorems for domains considerably more general than $\mathbb{R}^n_+$. We begin by describing the set of admissible domains. Let $B = \{(y_1, y_2, ..., y_n) \in \mathbb{R}^n : |y_j| < 1,\ 1 \leqslant j \leqslant n\}$, and set

$$B_+ = \{y \in B : y = (y', y_n) \text{ and } y_n > 0\}$$

and

$$B_0 = \{y \in B : y = (y', y_n) \text{ and } y_n = 0\}.$$

Here we continue to utilise the notation established in Section 2. In particular k is a fixed natural number throughout this section.

DEFINITION 4.1. A bounded, open set Ω in $\mathbb{R}^n$ with boundary $\partial\Omega$ will be called a V-domain if the following hold:

(V1) There exist open sets $G_1, ..., G_N \subset \mathbb{R}^n$ such that $\partial\Omega \subset \bigcup_{j=1}^N G_j$.

(V2) There exist locally $(k+1)$-smooth maps $\phi_j: \mathbb{R}^n \to \mathbb{R}^n$ such that $\phi_j(B) = G_j$, $\phi_j(B_+) = G_j \cap \Omega$, and $\phi_j(B_0) = G_j \cap \partial\Omega$, $j = 1, ..., N$.

(V3) Let Ω_δ be the set of all points in Ω whose distance from $\partial\Omega$ is less than δ. Then for some $\delta > 0$,

$$\Omega_\delta \subset \bigcup_{j=1}^N \phi_j\left(\left\{(y_1, y_2, ..., y_n) \in \mathbb{R}^n : |y_j| < \frac{1}{k+1}, 1 \leqslant j \leqslant n\right\}\right).$$

We continue to use the notations $|\cdot|_\Omega$ and $\|\cdot\|_\Omega$ as defined in Eqs. (4) and (5), as well as the space $X(\Omega)$. We now embark on the construction which will define our extension. So we presume Ω is a V-domain. We will define a linear extension operator $L: X(\Omega) \to X(\mathbb{R}^n)$. Let

$$Q = \left\{(y_1, y_2, ..., y_n) \in \mathbb{R}^n : |y_j| < \frac{1}{k+1}, 1 \leqslant j \leqslant n\right\}.$$

Now set $V_i = \phi_i(Q)$, $i = 1, ..., N$. By virtue of (V3) for some $\delta > 0$, $V_1, ..., V_N$ form an open cover of Ω_δ. Consequently, we can find an open set V_0 such that $\text{dist}(x, \partial\Omega) \geqslant \delta$ for all $x \in V_0$, and $\Omega \subset \bigcup_{j=0}^N V_j$. Now construct $u_0, ..., u_N \in C_0^\infty(\mathbb{R}^n)$ such that

(A1) each u_j is supported in V_j,

(A2) $u_j(x) \geqslant 0$ for all $x \in \mathbb{R}^n$,

(A3) $\sum_{j=0}^N u_j(x) = 1$ for all $x \in \Omega$.

Now take $f \in X(\Omega)$. Then $f = g|_\Omega$ for some $g \in C_0^k(\mathbb{R}^n)$ with $|g|_\Omega = |f|_\Omega < \infty$. Thus we can think of f as being in $C_0^k(\mathbb{R}^n)$. We can write

$$f(x) = \sum_{j=0}^N u_j(x) f(x) \qquad \text{for} \quad x \in \Omega.$$

Now define $\psi_j: \mathbb{R}^n \to \mathbb{R}$ by $\psi_j = (u_j f) \circ \phi_j$, $j = 1, ..., N$. Note that $(u_j f)(\phi_j(x)) = 0$ if $\phi_j(x) \notin V_j = \phi_j(Q)$. Hence ψ_j is supported on Q.

The notation we are developing here will be used throughout this section in the various results we shall establish, so the reader needs to have internalised the terminology to understand our subsequent arguments.

LEMMA 4.2. *Let $s \in C_0^k(\mathbb{R}^n)$ be supported on Q. Define $t = s|_{\mathbb{R}_+^n}$, and the extension operator E as in Definition 2.1. Then $Et \in C_0^k(\mathbb{R}^n)$ and is supported in B.*

Proof. The fact that $Et \in C_0^k(\mathbb{R}^n)$ is the substance of Theorem 2.2. To see that Et is supported in B, suppose $x \notin B$. If $x_n > 0$ then $(Et)(x) = t(x) = s(x) = 0$, since s is supported on Q and $Q \subset B$. If $x_n \leqslant 0$, then

$$Et(x) = \sum_{i=1}^{k+1} \lambda_i t(x', -x_n/i) = \sum_{i=1}^{k+1} \lambda_i s(x', -x_n/i).$$

Suppose $|x_n| \geqslant 1$. Then for $1 \leqslant i \leqslant k+1$,

$$|x_n/i| \geqslant \frac{1}{(k+1)} |x_n| \geqslant \frac{1}{(k+1)}.$$

If $|x_n| < 1$, then since $x \notin B$, there is a j with $1 \leqslant j \leqslant n-1$ such that

$$|x_j| \geqslant 1 \geqslant \frac{1}{k+1}.$$

From this we conclude that if $x \notin B$, then $(x', -x_n/i) \notin Q$ for $1 \leqslant i \leqslant k+1$. Hence, $(Et)(x) = 0$. ∎

Define $\Psi_j = \psi_j|_{\mathbb{R}^n_+}$. Then by Lemma 4.2, $E\Psi_j$ is in $C_0^k(\mathbb{R}^n)$ and is supported in B. Define $\theta_j = E\Psi_j \circ \phi_j^{-1}$. If $x \notin G_j$, it follows that $\phi_j^{-1}(x) \notin B$ and so $E\Psi_j(\phi_j^{-1}(x)) = 0$. From this we conclude that the support of θ_j is in G_j, $j = 1, ..., N$. We are now finally in a position to define our extension operator L as

$$Lf = u_0 f + \sum_{i=1}^{N} \theta_i. \tag{10}$$

LEMMA 4.3. *Let Ω be a V-domain. We have $Lf(x) = f(x)$ for all $x \in \Omega$.*

Proof. Take $x \in \Omega$. By reordering if necessary, we can assume that x belongs to $G_1, ..., G_M$ but not to $G_{M+1}, ..., G_N$. Thus,

$$\begin{aligned} Lf(x) &= u_0(x) f(x) + \sum_{i=1}^{M} \theta_i(x) \\ &= u_0(x) f(x) + \sum_{i=1}^{M} E\Psi_i(\phi_i^{-1}(x)). \end{aligned}$$

Now for $i = 1, ..., M$, $x \in \Omega \cap G_i$ and so $\phi_i^{-1}(x) \in B_+$. Hence,

$$E\Psi_i(\phi_i^{-1}(x)) = (u_i f)(\phi_i(\phi_i^{-1}(x))) = (u_i f)(x).$$

Finally, because $u_i(x)=0$, $i=M+1, \ldots, N$,

$$Lf(x)=u_0(x)\, f(x)+\sum_{i=1}^{M} u_i(x)\, f(x)=u_0(x)\, f(x)+\sum_{i=1}^{N} u_i(x)\, f(x)=f(x). \quad \blacksquare$$

Because of Lemma 4.3, L certainly has the potential to be the required extension operator. However, the main question is whether L is bounded. This question turns on the simple observation:

$$\|Lf\|_{\mathbb{R}^n} \leqslant \|u_0 f\|_{\mathbb{R}^n} + \sum_{j=1}^{N} \|\theta_j\|_{\mathbb{R}^n}.$$

The next result examines the quantities $\|\theta_j\|_{\mathbb{R}^n}$. We shall drop the subscript j temporarily and simply work with $\theta = E\Psi \circ \phi^{-1}$ supported on a set G, which typifies G_j.

Lemma 4.4. *Let Ω be a V-domain. Let w satisfy* (W1)–(W6). *There exists a number $C>0$ such that*

$$\|\theta\|_{\mathbb{R}^n} \leqslant C\, \|uf\|_{\Omega} \qquad \textit{for all} \quad f \in X(\Omega).$$

Proof. Let $f \in X(\Omega)$. For $\alpha \in \mathbb{Z}_+^n$, $|\alpha| \leqslant k$, we consider the integrals

$$I_1=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} w(x-y)\, |D^\alpha\theta(x)-D^\alpha\theta(y)|^2\, dx\, dy \quad \text{and} \quad I_2=\int_{\mathbb{R}^n} |D^\alpha\theta(x)|^2\, dx.$$

Let $\mathscr{G}$ be a bounded subset of $\mathbb{R}^n$ which contains G. Moreover, suppose there exists $\eta>0$ such that $|x-y|>\eta$ for all $x \in G$ and $y \in \mathbb{R}^n \setminus \mathscr{G}$. Then, because θ is supported on G, Lemma 3.7 provides a number c_1 such that

$$I_1 \leqslant c_1^2 \int_{\mathscr{G}}\int_{\mathscr{G}} w(x-y)\, |D^\alpha\theta(x)-D^\alpha\theta(y)|^2\, dx\, dy.$$

Again, because θ is supported on G,

$$I_2=\int_{\mathscr{G}} |D^\alpha\theta(x)|^2\, dx,$$

and so we conclude that $\|\theta\|_{\mathbb{R}^n} \leqslant c_1\, \|\theta\|_{\mathscr{G}}$. Since ϕ^{-1} is a locally $(k+1)$-smooth mapping, Theorem 3.13 shows there is a number $c_2>0$ such that $\|\theta\|_{\mathscr{G}}=\|E\Psi \circ \phi^{-1}\|_{\mathscr{G}} \leqslant c_2\, \|E\Psi\|_{\phi^{-1}(\mathscr{G})}$. Now, by Theorem 2.6, we can find a constant $c_3>0$ such that

$$\|\theta\|_{\mathbb{R}^n} \leqslant c_1 c_2\, \|E\Psi\|_{\phi^{-1}(\mathscr{G})} \leqslant c_1 c_2\, \|E\Psi\|_{\mathbb{R}^n} \leqslant c_3\, \|\Psi\|_{\mathbb{R}^n_+}.$$

Because Ψ is supported on $Q_+ \subset B \cap \mathbb{R}^n_+$, we can again apply Lemma 3.7 to obtain a constant c_4 such that

$$\int_{\mathbb{R}^n_+} \int_{\mathbb{R}^n_+} w(x-y)\, |D^\alpha \Psi(x) - D^\alpha \Psi(y)|^2 \, dx\, dy$$

$$\leqslant c_4 \int_{B_+} \int_{B_+} w(x-y)\, |D^\alpha \Psi(x) - D^\alpha \Psi(y)|^2 \, dx\, dy,$$

for all $\alpha \in \mathbb{Z}^n_+$ with $|\alpha| \leqslant k$. Therefore, there exists a constant c_5 such that

$$\|\theta\|_{\mathbb{R}^n} \leqslant c_3\, \|\Psi\|_{\mathbb{R}^n_+} \leqslant c_5\, \|\Psi\|_{B_+} = c_5\, \|\psi\|_{B_+}.$$

Moreover, since $\psi = (uf) \circ \phi$, an application of Theorem 3.13 shows that there is a constant c_6 such that

$$\begin{aligned} \|\theta\|_{\mathbb{R}^n} \leqslant c_5\, \|uf \circ \phi\|_{B_+} &\leqslant c_6\, \|uf\|_{\phi(B_+)} \\ &= c_6\, \|uf\|_{\Omega \cap G} \\ &\leqslant c_6\, \|uf\|_{\Omega}. \quad \blacksquare \end{aligned}$$

THEOREM 4.5. *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded V-domain. Let $w\colon \mathbb{R}^n \to \mathbb{R}$ be a measurable function satisfying* (W1)–(W6). *Let $f \in X(\Omega)$. Then there exists a continuous, linear mapping $L\colon X(\Omega) \to X(\mathbb{R}^n)$ such that for all $f \in X(\Omega)$,*

(1) $Lf|_\Omega = f$

(2) $\|Lf\|_{\mathbb{R}^n} \leqslant M\, \|f\|_\Omega$ *for some constant M independent of f.*

Proof. Let $f \in X(\Omega)$ and define Lf as in Eq. (10). By Lemma 4.3, $(Lf)(x) = f(x)$ for all $x \in \Omega$. Furthermore,

$$\|Lf\|_{\mathbb{R}^n} \leqslant \|u_0 f\|_{\mathbb{R}^n} + \sum_{j=1}^{N} \|\theta_j\|_{\mathbb{R}^n}.$$

An application of Lemma 4.4 shows that $\|\theta_j\|_{\mathbb{R}^n} \leqslant c_1\, \|u_j f\|_\Omega$ for some suitable constant $c_1 > 0$. Thus,

$$\|Lf\|_{\mathbb{R}^n} \leqslant \|u_0 f\|_{\mathbb{R}^n} + \sum_{j=1}^{N} c_1\, \|u_j f\|_\Omega.$$

An application of Lemma 3.15 gives

$$\|Lf\|_{\mathbb{R}^n} \leqslant \|u_0 f\|_{\mathbb{R}^n} + \sum_{j=1}^{N} c_1 c_2\, \|f\|_\Omega,$$

for some number c_2 independent of f. Furthermore, since u_0 is supported on $V_0 \subset \Omega$ we can use Lemma 3.7 and a further application of Lemma 3.15 to obtain constants $c_3, c_4 > 0$, independent of f, such that

$$\begin{aligned}\|Lf\|_{\mathbb{R}^n} &\leqslant c_3 \|u_0 f\|_\Omega + Nc_1c_2 \|f\|_\Omega \\ &\leqslant c_3c_4 \|f\|_\Omega + Nc_1c_2 \|f\|_\Omega \\ &\leqslant (c_3c_4 + Nc_1c_2) \|f\|_\Omega.\end{aligned}$$

Using this result and the fact that $f \in X(\Omega)$ we have

$$\begin{aligned}|Lf|_{\mathbb{R}^n} &\leqslant \|Lf\|_{\mathbb{R}^n} \leqslant (c_3c_4 + Nc_1c_2) \|f\|_\Omega \\ &= (c_3c_4 + Nc_1c_2)\left(\sum_{|\alpha| \leqslant k} \int_\Omega |D^\alpha f(x)|^2 \, dx + |f|_\Omega^2\right)^{1/2} < \infty.\end{aligned}$$

Thus $Lf \in X(\mathbb{R}^n)$. ∎

Let $\mathscr{X}(\Omega)$ be the completion of $X(\Omega)$ with respect to $\|\cdot\|_\Omega$. Let $\mathscr{Y}(\Omega)$ be the completion of $X(\Omega)$ with respect to $|\cdot|_\Omega$. We now apply our extension results to functions in $\mathscr{X}(\Omega)$ as follows.

THEOREM 4.6. *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded V-domain. Let $w\colon \mathbb{R}^n \to \mathbb{R}$ be a measurable function satisfying* (W1)–(W6). *There exists a continuous linear operator $\mathscr{L}\colon \mathscr{X}(\Omega) \to \mathscr{X}(\mathbb{R}^n)$ such that for all $f \in \mathscr{X}(\Omega)$,*

(1) $\mathscr{L}f|_\Omega = f$

(2) $\|\mathscr{L}f\|_{\mathbb{R}^n} \leqslant M \|f\|_\Omega$, *for some constant M independent of f.*

Proof. The result is derived from Theorem 4.5 using a standard abstract analysis result (see [7, **18.19**, p. 180]). ∎

We are able now to prove our final extension theorem.

THEOREM 4.7. *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded V-domain. Let $w\colon \mathbb{R}^n \to \mathbb{R}$ be a measurable function satisfying* (W1)–(W6). *Given $f \in \mathscr{Y}(\Omega)$, there exists a function $f_e \in \mathscr{Y}(\mathbb{R}^n)$ such that*

(1) $f_e|_\Omega = f$

(2) $|f_e|_{\mathbb{R}^n} \leqslant M |f|_\Omega$ *for some constant M independent of f.*

Proof. We shall work with the quotient space $\mathscr{X}(\Omega)/\Pi_k = \{f + \Pi_k : f \in \mathscr{X}(\Omega)\}$. For $f \in \mathscr{X}(\Omega)$ define

$$\begin{aligned}\|f + \Pi_k\|_1 &= |f|_\Omega, \\ \|f + \Pi_k\|_2 &= \inf\{|u|_{\mathbb{R}^n} : u \in \mathscr{Y}(\mathbb{R}^n) \text{ and } u|_\Omega = f\}.\end{aligned}$$

We claim that $\|\cdot\|_1$ and $\|\cdot\|_2$ are norms on $\mathscr{X}(\Omega)/\Pi_k$. Now, $|f|_\Omega = 0$ if and only if $f \in \Pi_k$, and so $\|\cdot\|_1$ is clearly a norm on $\mathscr{X}(\Omega)/\Pi_k$. Given $f \in X(\Omega)$, Theorem 4.5 allows us to find an $Lf \in X(\mathbb{R}^n)$ which satisfies $Lf|_\Omega = f$ and $|Lf|_{\mathbb{R}^n} < \infty$. Trivially, $Lf \in \mathscr{Y}(\mathbb{R}^n)$. As in Theorem 4.6, we can deduce that for each $f \in \mathscr{X}(\Omega)$, there is an $\mathscr{L}f \in \mathscr{Y}(\mathbb{R}^n)$ which satisfies $\mathscr{L}f|_\Omega = f$ and $|\mathscr{L}f|_{\mathbb{R}^n} < \infty$. Hence, $\|f + \Pi_k\|_2$ exists. Let $f_e \in \mathscr{Y}(\mathbb{R}^n)$ satisfy $|f_e|_{\mathbb{R}^n} = \inf\{|u|_{\mathbb{R}^n}: u \in \mathscr{Y}(\mathbb{R}^n)$ and $u|_\Omega = f\}$. Suppose $\|f + \Pi_k\|_2 = 0$, then $|f_e|_{\mathbb{R}^n} = 0$ and $f_e \in \Pi_k$. Since $f_e|_\Omega = f$ this implies $f \in \Pi_k$. Conversely, suppose $f \in \Pi_k$. Then f_e is just the polynomial in Π_k for which $f_e|_\Omega = f$, since then $|f_e|_{\mathbb{R}^n} = 0$. Hence $\|\cdot\|_2$ is a norm on $\mathscr{X}(\Omega)/\Pi_k$.

The quotient map $Q: \mathscr{X}(\Omega) \to \mathscr{X}(\Omega)/\Pi_k$ is defined by $Q(f) = f + \Pi_k$, for $f \in \mathscr{X}(\Omega)$. This is a linear, continuous, open map from $\mathscr{X}(\Omega)$ to $\mathscr{X}(\Omega)/\Pi_k$, (see for example [14, p. 31]). Since $\mathscr{X}(\Omega)$ is complete we can thus deduce that the normed spaces $(\mathscr{X}(\Omega)/\Pi_k, \|\cdot\|_1)$ and $(\mathscr{X}(\Omega)/\Pi_k, \|\cdot\|_2)$ are also complete. For details see [7, **18.16**, p. 179]. For all $f \in \mathscr{X}(\Omega)$, we have the simple inequality

$$\|f + \Pi_k\|_1 = |f|_\Omega = |f_e|_\Omega \leqslant |f_e|_{\mathbb{R}^n} = \|f + \Pi_k\|_2.$$

Hence, using standard Banach space theory [7, Corollary 22.12, p. 218], there exists a $\beta > 0$ such that

$$|f_e|_{\mathbb{R}^n} = \|f + \Pi_k\|_2 \leqslant \beta \, \|f + \Pi_k\|_1 = \beta \, |f|_\Omega, \qquad \text{for all} \quad f \in \mathscr{X}(\Omega).$$

A consequence is that for each $f \in X(\Omega)$, we can find $f_e \in \mathscr{Y}(\mathbb{R}^n)$ such that $f_e|_\Omega = f$ and $|f|_{\mathbb{R}^n} \leqslant \beta \, |f|_\Omega$. Another application of [7, **18.19**, p. 180] completes the proof. ∎

5. THE WEIGHT FUNCTION w AND THE DOMAIN Ω

The extension results developed in the previous section are dependent on the weight function w satisfying conditions (W1)–(W6), as given in Section 3. We give now some examples of weight functions for which these properties hold.

We begin with the familiar non-integer-valued Sobolev seminorms. Here the weight function w is defined by $w(x) = |x|^{-n-\lambda}$ for $x \in \mathbb{R}^n$ and $0 < \lambda < 2$. It is clear that w satisfies conditions (W1)–(W4) and (W6). To see that (W5) is satisfied, let ϕ be a locally 1-smooth map on $\mathbb{R}^n$. Then ϕ^{-1} is also locally 1-smooth. Let Ω be a bounded domain. By Taylor's formula, there exists a constant $K > 0$ such that for all $x, y \in \Omega$,

$$|x - y| = |\phi^{-1}(\phi(x)) - \phi^{-1}(\phi(y))| \leqslant K \, |\phi(x) - \phi(y)|.$$

Hence, for all $x, y \in \Omega$ with $x \neq y$,

$$w(\phi(x)-\phi(y)) = \frac{1}{|\phi(x)-\phi(y)|^{n+\lambda}} \leqslant K^{n+\lambda} \frac{1}{|x-y|^{n+\lambda}} = K^{n+\lambda} w(x-y).$$

Since ϕ is a bijection, $x=y$ implies $w(\phi(x)-\phi(y)) = w(x-y) = w(0)$. Hence, $w(\phi(x)-\phi(y)) \leqslant \max\{K^{n+\lambda}, 1\}\, w(x-y)$ for all $x, y \in \Omega$. Hence, condition (W5) is satisfied.

For our second example let $w(x) = e^{-|x|^2}$ for $x \in \mathbb{R}^n$. Again it is easily verified that w satisfies conditions (W1)–(W4) and (W6). Let ϕ be a locally smooth map on $\mathbb{R}^n$. Let Ω be a bounded domain. For all $x, y \in \Omega$,

$$|x-y|^2 - |\phi(x)-\phi(y)|^2 \leqslant |x-y|^2 \leqslant \sup_{x, y \in \Omega} |x-y|^2.$$

Because Ω is bounded we can find a $K > 0$ such that $\sup_{x, y \in \Omega} |x-y|^2 \leqslant K$. Thus,

$$|\phi(x)-\phi(y)|^2 \geqslant |x-y|^2 - K, \qquad \text{for all} \quad x, y \in \Omega.$$

Thus, for all $x, y \in \Omega$,

$$w(\phi(x)-\phi(y)) = e^{-|\phi(x)-\phi(y)|^2} \leqslant e^{-|x-y|^2+K} = e^K w(x-y).$$

Consequently, condition (W5) holds.

Our previous example forms part of a family of such examples. Let w be a continuous, positive-valued function in $L^1(\mathbb{R}^n)$ satisfying $w(-x) = w(x)$ for all $x \in \mathbb{R}^n$. We also assume that there exists some ball $B_\delta = \{x \in \mathbb{R}^n : |x| \leqslant \delta\}$ such that on $\mathbb{R}^n \setminus B_\delta$, $w(x)$ is a decreasing function of $|x|$. It is straightforward to see that w satisfies (W2)–(W4). Furthermore, there exists $A > 0$ such that $w(x) \leqslant A$ for all $x \in \mathbb{R}^n$. Let ϕ be a locally smooth map on $\mathbb{R}^n$ and let Ω be a bounded domain. Since w is continuous we can find $M > 0$ such that $w(x-y) \geqslant M$ for all $x, y \in \Omega$. Thus

$$w(\phi(x)-\phi(y)) \leqslant A \leqslant \frac{A}{M} w(x-y) \qquad \text{for all} \quad x, y \in \Omega.$$

Hence, (W5) is satisfied. Finally, we examine condition (W6). Take $y \in \mathbb{R}^n$. If $y \in B_\delta$ then a similar argument to that above proves the existence of $C > 0$ such that $w(y) \geqslant C w(x)$ for all $x \in \mathbb{R}^n$. If $y \notin B_\delta$, then $w(y) \geqslant w(x)$ for all $x \in \mathbb{R}^n$ with $|x| \geqslant |y|$. Thus $w(y) \geqslant \min\{1, C\}\, w(x)$ whenever $|x| > |y|$, showing (W6) holds.

The condition on the domain is more difficult to exemplify. If Ω is a domain which lies locally on one side of its boundary $\partial\Omega$, then Conditions

($V1$) and ($V2$) in Definition 4.1 will hold if the boundary $\partial\Omega$ is an $(n-1)$-dimensional, $(k+1)$-smooth manifold in $\mathbb{R}^n$. An easy example of a set Ω in $\mathbb{R}^2$, which satisfies ($V3$) is given by any disc. To construct the open sets G_j for the disc $B(0, r) = \{x \in \mathbb{R}^2 : |x| < r\}$ we can take

$$G_j = \left\{ x \in \mathbb{R}^2 : x = (\rho \cos \theta, \rho \sin \theta) \text{ and } \frac{7r}{8} < \rho < \frac{9r}{8}, \frac{(j-1)\pi}{8} < \theta < \frac{(j+1)\pi}{8} \right\}, \qquad j = 1, ..., 8.$$

The condition that Ω is a V-domain is essentially a fairly strong requirement on the smoothness of the boundary $\partial\Omega$. For example, this condition implies the strong local Lipschitz property, the uniform cone property, and the segment property (see Adams [1] for the appropriate definitions).

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